Irreducible p-modular representations of U(2,1)

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Abstract

Let E/F be a unramified quadratic extension of non-archimedean local fields of odd characteristic p, and G be the unramified unitary group $U(2,1)(E/F)$. For an irreducible smooth representation π of G over $\overline{\mathbf{F}}_p$, with an underlying irreducible smooth representation σ of a maximal compact open subgroup K, we prove that π admits eigenvectors for an appropriate Hecke operator T_σ, and we classify those π with non-zero eigenvalues for T_σ by a tree argument; as a corollary, we show π is supersingular if and only if it is supercuspidal.

Contents

1 Introduction

In recent years, there are great interests to study p-modular representations of p-adic reductive groups, as mainly motivated by Serre's modularity conjecture (and its generalizations) and by a potential mod-p local Langlands correspondence. Starting from the pioneering work of Barthel-Livné in 1990s ([BL95], [BL94]), much progress has been made during the last two decades to classify those irreducible admissible representations arising from parabolic induction, but still very little is known about the remaining mysterious supersingular representations, except the group $GL_2(\mathbf{Q}_p)$ ([Bre03]) and some other groups closely related it ([Abd14], [Koz16]), in which cases the supersingular representations are fully understood. For many related topics and backgrounds, and many previous results in this area (up to the summer of 2010) , we refer the readers to Breuil's ICM report ([Bre10]). For an overview of some current developments in this area, the readers should refer [Har16].

We start to describe the main results in this paper. Let E/F be a unramified quadratic extension of non-archimedean local fields of odd characteristic p, and G be the unitary group $U(2,1)(E/F)$ in three variable. In this paper, we investigate irreducible smooth representations of G on a vector space over $\overline{\mathbf{F}}_p$.

Let $B = HN$ be the standard Borel subgroup of G, in which H is the diagonal subgroup of G and N is the upper unipotent radical of B. Let π be an irreducible smooth representation of G, containing an irreducible smooth representation σ with respect to a maximal compact open subgroup K. The representation σ has a unique line $\sigma^{I_{1,K}}$, consisting of invariants for the action of pro-p Iwahori subgroup $I_{1,K}$; the Iwahori subgroup I_K acts on that line by a character χ_σ. The spherical Hecke algebra $\mathcal{H}(K,\sigma)$ is a polynomial ring in one variable, i.e., $\mathcal{H}(K,\sigma) \cong \overline{\mathbf{F}}_p[T_\sigma]$, for some $T_\sigma \in \mathcal{H}(K,\sigma)$ (Definition 3.8). The space $\mathrm{Hom}_G(\mathrm{ind}_K^G\sigma,\pi)$ is shown to admit an eigenvector for the natural action of $\mathcal{H}(K,\sigma)$ (Theorem 1.2), and let $\lambda \in \overline{\mathbf{F}}_p$ be the corresponding eigenvalue for T_σ. The main result of this paper classifies those π with *non-zero* λ:

Theorem 1.1. *((2) of Theorem 3.1)*

Assume that $\lambda \neq 0$. Set a character ε of B: $\varepsilon\mid_{H\cap K} = \chi_\sigma^s$[1], and $\varepsilon(\alpha) = \lambda$.

(1). *If χ_σ does not factor through determinant, or $\lambda \neq 1$, then*

$$\pi \cong \mathrm{ind}_B^G\,\varepsilon.$$

[1]Here, χ^s is the conjugate of χ, see Remark 2.2.

(2). *If χ_σ factors through determinant, i.e., $\chi_\sigma = \eta \circ \det$ for some character of $E^\times$, and $\lambda = 1$. Then*

$$\pi \cong \begin{cases} \eta \circ \det, & \text{if } \dim \sigma = 1, \\ \eta \circ \det \otimes St, & \text{otherwise.} \end{cases}$$

Here, St is the Steinberg representation of G, defined as $\mathrm{ind}_B^G 1/1$.

In our Theorem 1.1, the representation π is not assumed to be admissible; if that was assumed, the statements in Theorem 1.1 could be read out from the recent breakthrough work of Abe-Henniart-Herzig-Vignéras ([AHHV14]). However, we have the following result, which allow us to remove such an assumption.

Theorem 1.2. *((1) of Theorem 3.1) The space*

$$\mathrm{Hom}_G(\mathrm{ind}_K^G \sigma, \pi)$$

admits eigenvectors from the action of the spherical Hecke algebra $\mathcal{H}(K, \sigma)$.

In view of Theorem 1.1, we put the following:

Definition 1.3. *An irreducible smooth representation π of G is called supersingular if it is a quotient of $\mathrm{ind}_K^G \sigma/(T_\sigma)$, for some irreducible smooth representation σ of K.*

The following corollary results from Theorem 1.1:

Corollary 1.4. *An irreducible smooth representation π is supersingular if and only if π is supercuspidal.*

We end this introduction with a brief discussion on the novelty of our results and compare the approach we have taken with those in literatures. First of all, this paper originates from both authors' PhD thesis ([Abd11], [Xu14]), and it collects most results we have at hand on the group G. Secondly, for a connected reductive group $\mathbf{G}$ defined over F, the recent work of Abe-Henniart-Herzig-Vignéras ([AHHV14]) has reduced the classification of irreducible admissible p-modular representations of $G = \mathbf{G}(F)$ to the so-called supersingular representations[2]; especially they have proved (that in this case) supersingularity is equivalent to supercuspidality. The main ingredients involved in [AHHV14] are mainly the theory of Satake isomorphism and pro-p Iwahori-Hecke algebra, developed by these authors in a series of

[2]In [AHHV14, I.5], an irreducible admissible representation of G is called supersingular if all its (Hecke) eigenvalues are supersingular.

papers, just to mention some of them: [Her11b], [Her11a], [Abe13], [HV12], [HV15], [Vig16].

In contrast, the results we prove here are more modest, but there are differences. The first one, as already mentioned before, we don't need the assumption of admissibility of irreducible smooth representations, for which we have a substitute, that is Theorem 1.2. Note that in [AHHV14], admissibility of irreducibility smooth representations is a fundamental assumption, see the discussion in their III.26. On the other hand, we expect our Theorem 1.2 would have some other applications; it is the counterpart of a theorem of Barthel–Livné on $GL_2(F)$, and the latter has been used crucially in some recent work, for examples in [Hu12] and in [Ber12]. The second difference is that we use a tree argument to prove Theorem 1.1, following the approach of Barthel–Livné. As the group $G = U(2,1)(E/F)$ has two maximal compact open subgroups up to conjugacy, it causes extra complication to carry out the strategy.

The paper is organized as follows. In section 2, we fix notations and recall some preliminary facts which will be used throughout the paper. In section 3, we prove most parts of Theorem 1.1. In the last section 4, we prove Theorem 1.2 in full, which completes the proof of Theorem 1.1.

2 Notations and Preliminary

2.1 Notations

Let F be a non-archimedean local field, with ring of integers $\mathfrak{o}_F$ and maximal ideal $\mathfrak{p}_F$, and k_F be its residue field of odd cardinality $q = p^f$. Fix a separable closure F_s of F. Let E be the unramified quadratic extension of F in F_s. We use similar notations $\mathfrak{o}_E$, $\mathfrak{p}_E$, k_E for E. Denote by E^1 (resp, k_E^1) the subgroup of $E^\times$ (resp, $k_E^\times$) of elements of norm 1. Let ϖ_E be a prime element of E, lying in F. Given a 3-dimensional vector space V over E, we identify it with E^3, by fixing a basis of V. Equip V with the non-degenerate Hermitian form h:

$$\mathrm{h}:\ V \times V \to E,\ (v_1, v_2) \mapsto\ v_1^{\mathrm{T}} \beta \overline{v_2}, v_1, v_2 \in V.$$

Here, $-$ is the non-trivial conjugate in $\mathrm{Gal}(E/F)$, and β is the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The unitary group G is defined as:

$$G=\{g\in \mathrm{GL}(3,E):\mathrm{h}(gv_1,gv_2)=\mathrm{h}(v_1,v_2),\text{for any } v_1,v_2\ \in V\}.$$

Let $B=HN$ be the subgroup of upper triangular matrices of G, where N is the unipotent radical of B and H is the diagonal subgroup of G. Denote an element of the following form in N by $n(x,y)$:

$$\begin{pmatrix}1 & x & y\\ 0 & 1 & -\bar{x}\\ 0 & 0 & 1\end{pmatrix}$$

where $(x,y)\in E^2$ satisfies $x\bar{x}+y+\bar{y}=0$.

Denote by N_k, for any $k\in\mathbb{Z}$, the subgroup of N consisting of $n(x,y)$ with $y\in\mathfrak{p}_E^k$.

Up to conjugacy, the group G has two maximal compact open subgroups K_0 and K_1, which explicitly are

$$K_0=\begin{pmatrix}\mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{o}_E\\ \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{o}_E\\ \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{o}_E\end{pmatrix}\cap G,\ K_1=\begin{pmatrix}\mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-1}\\ \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{o}_E\\ \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{o}_E\end{pmatrix}\cap G$$

Let α be the matrix

$$\begin{pmatrix}\varpi_E^{-1} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \varpi_E\end{pmatrix},$$

and put $\beta'=\beta\alpha^{-1}$. Note that $\beta\in K$ and $\beta'\in K'$.

Let I be the standard Iwahori subgroup of G consisting of matrices which are upper triangular mod $\mathfrak{p}_E$; it is the intersection of K_0 and K_1. Denote by I_1 the pro-p Sylow subgroup of I. Put $H_0=I\cap H$, $H_1=I_1\cap H$.

We have introduced several subgroups of G, say B,N,N_k, and later on we will use the notations B',N',N'_k for their conjugate subgroups of G by the element β. Also, we use the notation $n'(x,y)$ for the element in N':

$$\begin{pmatrix}1 & 0 & 0\\ x & 1 & 0\\ y & -\bar{x} & 1\end{pmatrix}.$$

For a maximal compact open subgroup K, which is conjugate to K_0 or K_1, the quotient $K/K(1)$ by its first congruence subgroup $K(1)$ is (canonically) isomorphic to a finite group Γ_K, i.e., $U(2,1)(k_E/k_F)$ or $U(1,1)\times U(1)(k_E/k_F)$. When K is K_0 or K_1, denote the element β or β' in K by β_K. Denote by I_K (resp, I'_K) the inverse image of upper-triangular subgroup (resp, lower triangular subgroup) of Γ_K in K, and $I_{1,K}$ (resp, $I'_{1,K}$) be the

pro-p Sylow subgroup of I_K (resp, I'_K). There is a unique integer n_K and m_K, such that $I_K \cap N = N_{n_K}$ and $I_K \cap N' = N'_{m_K}$.

For an irreducible smooth representation σ of K, and for a $g \in G$, $v \in \sigma$, as usual we denote by $[g, v]$ the function in the compact induction $\mathrm{ind}_K^G \sigma$, which is supported on Kg^{-1} and takes value v at g^{-1}.

2.2 Preliminary facts

Proposition 2.1. *[Abd11]* (1) $G = BK_i$, *for* $i = 0, 1$.

(2) $G = \cup_{l\geq 0} K_i \alpha^l K_i$, *for* $i = 0, 1$.

(3) $K_0 = I \cup I\beta I,\ K_1 = I \cup I\beta' I$.

For $y \neq 0$, the following equality will be used repeatedly

$$\beta n(x, y) = n(\bar{y}^{-1}x, y^{-1}) \cdot \mathrm{diag}(\bar{y}^{-1}, -\bar{y}y^{-1}, y) \cdot n'(-\bar{y}^{-1}\bar{x}, y^{-1}). \quad (1)$$

Remark 2.2. *Let χ be a character of the group H_0. Write χ as $\chi_1 \otimes \chi_2$, i.e., $\chi(\mathrm{diag}(x, y, \bar{x}^{-1})) = \chi_1(x)\chi_2(y)$, for $\mathrm{diag}(x, y, \bar{x}^{-1}) \in H_0$, where χ_1 and χ_2 are respectively characters of $k_E^\times$ and k_E^1. Denote by χ^s the conjugate of χ, explicitly $\chi(\mathrm{diag}(x, y, \bar{x}^{-1})) = \chi_1(\bar{x}^{-1}))\chi_2(y)$. Hence, $\chi = \chi^s$ is equivalent to χ_1 being trivial on the group $k_F^\times$; it is equivalent to the existence of a (unique) character $\bar{\chi}_1$ of k_E^1, such that $\chi_1(x) = \bar{\chi}_1(x\bar{x}^{-1})$. Furthermore, χ factors through the determinant if and only if $\chi_2 = \bar{\chi}_1$.*

2.3 The Bruhat-Tits tree of G

In this part, we record (actually prove) some facts about the Bruhat-Tits tree of the group G, which will be used essentially later.

Let $\triangle$ be the tree associated to G. Denote by X_0 the set of vertices on $\triangle$, which consists of $\mathfrak{o}_E$-lattices $\mathcal{L}$ in E^3, such that

$$\varpi_E \mathcal{L} \subset \mathcal{L}^* \subset \mathcal{L},$$

where $\mathcal{L}^*$ is the dual lattice of $\mathcal{L}$, under the Hermitian form h, i.e., $\mathcal{L}^* = \{v \in V : h(v, \mathcal{L}) \in \mathfrak{p}_E\}$.

Let $\mathbf{v}$, $\mathbf{v}'$ be two vertices in X_0 represented by $\mathcal{L}$ and $\mathcal{L}'$. The vertices $\mathbf{v}$ and $\mathbf{v}'$ are said to be adjacent, if:

$$\mathcal{L}' \subset \mathcal{L} \text{ or } \mathcal{L} \subset \mathcal{L}'.$$

Let $\{e_{-1}, e_0, e_1\}$ be the standard basis of E^3. Consider the following two lattices in E^3:

$$\mathcal{L}_0 = \mathfrak{o}_E e_{-1} \oplus \mathfrak{o}_E e_0 \oplus \mathfrak{o}_E e_1,\ \ \mathcal{L}_1 = \mathfrak{o}_E e_{-1} \oplus \mathfrak{o}_E e_0 \oplus \mathfrak{p}_E e_1.$$

Denote respectively by $\mathbf{v}_0, \mathbf{v}_1$ the two vertices represented by $\mathcal{L}_0$ and $\mathcal{L}_1$, which are then adjacent. The natural action of G on X_0 consists of two orbits:

$$X_0 = \{G \cdot \mathbf{v}_0\} \cup \{G \cdot \mathbf{v}_1\}.$$

The stabilizers of $\mathbf{v}_0$ and $\mathbf{v}_1$ in G are respectively K_0 and K_1.

For an integer $n \in \mathbb{Z}$, put $\mathbf{v}_{2n} = \alpha^n \mathbf{v}_0, \mathbf{v}_{2n+1} = \alpha^n \mathbf{v}_1$. These vertices together form a standard apartment A in $\triangle$: $\{\mathbf{v}_n, n \in \mathbb{Z}\}$. A general edge in this apartment is $e_{2n,2n\pm 1} = (\mathbf{v}_{2n}, \mathbf{v}_{2n\pm 1})$, for an integer $n \in \mathbb{Z}$, i.e., an edge with origin $\mathbf{v}_{2n}$ and terminus $\mathbf{v}_{2n\pm 1}$. Note that the stabilizer of the edge $e_{0,1}$ is just the Iwahori subgroup $I = K_0 \cap K_1$.

Denote by ∞ the positive end of this standard apartment. For any vertex $\mathbf{v}$, let $\overline{\mathbf{v}\infty}$ be the geodesic ray from $\mathbf{v}$ to ∞. So we can find an integer k such that $\mathbf{v}_k \in \overline{\mathbf{v}\infty}$. Define the *height* $h(\mathbf{v})$ of $\mathbf{v}$ as $k - d(\mathbf{v}_k, \mathbf{v})$. Note that this definition is independent of the choice of k and that $h(\mathbf{v}_k) = k$.

Given any two vertices $\mathbf{v}$ and $\mathbf{v}'$, *we say* $\mathbf{v}$ *is under* $\mathbf{v}'$, *if* $\mathbf{v}' \in \overline{\mathbf{v}\infty}$. The following two lemmas are useful:

Lemma 2.3. $(N/N_{-r})\mathbf{v}_r = \{\mathbf{v} \in X_0 : \ h(\mathbf{v}) = r\}$

Proof. Firstly, we note that the stabilizers of $\mathbf{v}_{2k}$ and $\mathbf{v}_{2k+1}$ in G are respectively $\alpha^k K_0 \alpha^{-k}$ and $\alpha^k K_1 \alpha^{-k}$ for any integer k. Therefore the stabilizers of $\mathbf{v}_{2k}$ and $\mathbf{v}_{2k+1}$ in N are respectively $N \cap \alpha^k K_0 \alpha^{-k}$ and $N \cap \alpha^k K_1 \alpha^{-k}$. But these are exactly N_{-2k} and $N_{-(2k+1)}$.

Secondly, we are going to show: for a non-negative integer l and an integer r, $(N_{-(r+l)}/N_{-r})\mathbf{v}_r = \{\mathbf{v} \in X_0 : \ h(\mathbf{v}) = r, \ \mathbf{v}_{r+l} \in \overline{\mathbf{v}\infty}\}$. There are two steps:

Step 1 For $u \in N$, $u\mathbf{v}_r$ is also of height r. Take an integer k such that u fixes $\mathbf{v}_k$ and $\mathbf{v}_k \in \overline{u\mathbf{v}_r\infty}$. So by definition, $h(u\mathbf{v}_r) = k - d(\mathbf{v}_k, u\mathbf{v}_r)$, which equals r by the choice of k. As we have fixed ∞, we understand that $u\mathbf{v}_{r+l} \in \overline{u\mathbf{v}_r\infty}$ for all non-negative integers l. In particular, when we restrict to $u \in N_{-(r+l)}$, we get $\mathbf{v}_{r+l} \in \overline{u\mathbf{v}_r\infty}$. We have shown that $(N_{-(r+l)}/N_{-r})\mathbf{v}_r$ is contained in $\{\mathbf{v} \in X_0 : \ h(\mathbf{v}) = r, \ \mathbf{v}_{r+l} \in \overline{\mathbf{v}\infty}\}$.

Step 2. For a non-negative integer l and an integer r, denote respectively by n_r^l and m_r^l the cardinality of $N_{-(r+l)}/N_{-r}$ and that of the set M_r^l, where M_r^l is $\{\mathbf{v} \in X_0 : \ h(\mathbf{v}) = r, \ \mathbf{v}_{r+l} \in \overline{\mathbf{v}\infty}\}$. The list for n_r^l is as follows:

$$n_r^l \ = \begin{cases} q^{2l}, & \text{if } l \text{ is even,} \\ q^{2l+(-1)^{r-1}}, & \text{if } l \text{ is odd.} \end{cases} \tag{2}$$

To see this, we reduce the above to two special cases by conjugating by some power of α: n_{-l}^l and n_{1-l}^l, namely the cardinality of N_0/N_l and N_{-1}/N_{l-1}.

We deal with n_{-l}^l in detail. Given an even l, we have $n_{-l}^l = (n_{-2}^2)^{\frac{l}{2}}$. But $n_{-2}^2 = n_{-1}^1 \cdot n_{-2}^1 = q^3 \cdot q = q^4$. So in this case $n_{-l}^l = q^{2l}$. When l is odd, $n_{-l}^1 = n_{-1}^1 = q^3$. Now $l-1$ is even, and from the even case we get $n_{-l}^l = q^{2(l-1)} \cdot q^3 = q^{2l+1}$. Similarly we can show n_{1-l}^l as required in (2).

To compute m_r^l, we firstly note that there exists an induction relation between them by observing the tree: $m_r^{l+1} = m_r^l \cdot c_{r+l+1}$, where we denote by c_t the number of vertices adjacent to and under v_t. We know that it equals q or q^3, depending on whether t is odd or not. So we only need to compute some initial cases. The result is: $m_r^0 = 1$ for any r, $m_r^1 = q$ or q^3, depending on whether r is even or not. Combining the initial cases and the induction relation, we have finally shown that m_r^l is exactly given by the formula in (2). $\square$

Definition 2.4. *For a vertex $\mathbf{v} \in X_0$ and a positive integer n, the n-*antecedent *$a^n(\mathbf{v})$ of $\mathbf{v}$ is the unique vertex of height $h(\mathbf{v}) + n$ which is of distance n from $\mathbf{v}$.*

Lemma 2.5. *$a^l(u\mathbf{v}_k) = u\mathbf{v}_{k+l}$ for all positive integers l and all $k \in \mathbb{Z}$, and all $u \in N$.*

Proof. The l-antecedent of $\mathbf{v}_k$ is $\mathbf{v}_{k+l}$ by definition above. As the action of N preserves height (as observed in the proof of Lemma 2.3) and distance, we are done. $\square$

2.4 Spherical Hecke algebra $\mathcal{H}(K, \sigma)$ and the Hecke operator T

Let K be a maximal compact open subgroup of G. It is known that $\sigma^{I_{1,K}}$ and $\sigma_{I'_{1,K}}$ are both one-dimensional ([CE04, Theorem 6.12]), and that the natural composition map $\sigma^{I_{1,K}} \hookrightarrow \sigma \twoheadrightarrow \sigma_{I'_{1,K}}$ is non-zero, i.e., an isomorphism of vector spaces. Denote by j_σ the inverse of the former composition map. Especially, $j_\sigma(\bar{v}) = v$, for $v \in \sigma^{I_{1,K}}$, and it vanishes on $\sigma(I'_{1,K})$. As a result, there is a unique constant $\lambda_{\beta_K,\sigma} \in \overline{\mathbf{F}}_p$, satisfying that $\beta_K \cdot v - \lambda_{\beta_K,\sigma} v \in \sigma(I'_{1,K})$. for any $v \in \sigma^{I_{1,K}}$.

Remark 2.6. *$\lambda_{\beta_K,\sigma}$ is zero unless σ is a one-dimensional character ([HV12, Proposition 3.12]).*

Denote by $\mathcal{H}(K, \sigma)$ the spherical Hecke algebra $\mathrm{End}_G(\mathrm{ind}_K^G \sigma)$.

Proposition 2.7. *The algebra $\mathcal{H}(K, \sigma)$ is a polynomial ring in one variable, i.e., there is a $T \in \mathcal{H}(K, \sigma)$ so that:*

$$\mathcal{H}(K,\sigma) \cong \overline{\mathbf{F}}_p[T]$$

Proof. This was firstly proved in [Abd11, Theorem 4.5.2], using Satake transform. See also [Xu14, Proposition 3.3] for an explicit determination of the algebraic structure. □

Remark 2.8. *When K is K_0 and F is characteristic 0, Proposition 2.7 is a special case of a general theorem due to Herzig ([Her11b, Corollary 1.3]). The Hecke operator T in the statement of last Proposition is chosen in a way that, for example when $\sigma = 1$, it maps the function 1_K to the function $1_{K\alpha K}$, i.e., it sums over the neighbourhoods of distance two around $\mathbf{v}_0$. In general, we will recall its definition in the following part. Before doing that, it is worth pointing out that such T is not always identical to the operator that arises from Satake isomorphism ([Her11b], [Abd11]).*

We recall briefly how the Hecke operator T in the above Proposition is defined. By [BL94, Proposition 5], the algebra $\mathcal{H}(K,\sigma)$ is isomorphic to $\mathcal{H}_K(\sigma)$, which is the convolution algebra consisting of all functions f from G to the space $\mathrm{End}(\sigma)$, with compact support, satisfying $f(kgk') = \sigma(k)f(k)\sigma(k')$ for any $k, k' \in K, g \in G$. Let T be the operator which corresponds to the function $\varphi \in \mathcal{H}_K(\sigma)$, supported on $K\alpha K$, and satisfying $\varphi(\alpha) = j_\sigma$.

Let v be a vector in V, and hence from [BL94, (8)] that,

$$T\,[Id, v] = \sum_{g\in G/K} [g, \varphi(g^{-1})\cdot v]. \tag{3}$$

The function φ is supported on the double coset $K\alpha K$ $(= K\alpha^{-1}K)$, and we decompose $K\alpha^{-1}K$ into right cosets of K: $K\alpha^{-1}K = \cup_{k\in K/K\cap\alpha^{-1}K\alpha}\, k\alpha^{-1}K$, where k goes through $N_{n_K+1}/N_{n_K+2} \cup \beta_K \cdot N_{n_K}/N_{n_K+2}$. Then the formula of T ((3)) becomes:

$$T\,[Id, v] = \sum_{u\in N_{n_K+1}/N_{n_K+2}} [u\alpha^{-1}, j_\sigma v] + \sum_{u\in N_{n_K}/N_{n_K+2}} [\beta_K u\alpha^{-1}, j_\sigma\sigma(u^{-1}\beta_K)v]. \tag{4}$$

3 Comparison of compact induction with principal series

Let σ be an irreducible smooth representation of K, and the Iwahori subgroup I_K in K acts as a character on the line $\sigma^{I_{1,K}}$, for which we denote by χ_σ.

The results of this section are summarized in the following theorem.

Theorem 3.1. *Let π be an irreducible smooth representation of G and σ be an irreducible sub-representation of $\pi|_K$. Then,*

(1). *The space*

$$\mathrm{Hom}_G(\mathrm{ind}_K^G\sigma,\ \pi)$$

has an eigenvector for the action of the Hecke algebra $\mathcal{H}(K, \sigma)$.

(2). *Let λ be an eigenvalue of T in* (1). *Assume further that:*

$$\lambda \neq \begin{cases} -\bar{\chi}_1(-1), & \text{if } \chi_\sigma = \chi_\sigma^s = \chi_1 \otimes \chi_2, \\ 0, & \text{otherwise.} \end{cases}$$

We set a character ε of B such that $\varepsilon|_{H_0} = \chi_\sigma^s$, and

$$\varepsilon(\alpha) = \begin{cases} \lambda + \bar{\chi}_1(-1), & \text{if } \chi_\sigma = \chi_\sigma^s, \\ \lambda, & \text{otherwise.} \end{cases}$$

Then, we have the following,

(*a*). *If χ_σ does not factor through determinant, or $\lambda \neq 1 - \bar{\chi}_1(-1)$, then we have*

$$\pi \cong \mathrm{ind}_B^G\, \varepsilon.$$

(*b*). *If χ_σ factors through the determinant, i.e., $\chi_\sigma = \eta \circ \det$ for some character of k_E^1, and $\lambda = 1 - \bar{\chi}_1(-1)$. Then*

$$\pi \cong \begin{cases} \eta \circ \det, & \text{if } \dim \sigma = 1, \\ \eta \circ \det \otimes St, & \text{otherwise.} \end{cases}$$

Here, St is the Steinberg representation of G, defined as $\mathrm{ind}_B^G 1/1$.

3.1 Irreducibility of principal series and its eigenvalues

Theorem 3.2. (1). *For a character ε of B, the principal series $\mathrm{ind}_B^G\, \varepsilon$ is irreducible if and only if ε does not factor through the determinant.*

(2). *When $\varepsilon = \chi \circ \det$, the principal series $\mathrm{ind}_B^G\, \varepsilon$ is of length two, and it is the non-split extension of the $\chi \circ \det \otimes St$ by the one-dimensional character $\chi \circ \det$.*

(3). *Any irreducible smooth non-supercuspidal representation of G is isomorphic to a unique representation appearing in* (1) *and* (2).

Remark 3.3. *These results were firstly proved in the first named author's thesis ([Abd11, Theorem 4.1.3]). In its most generality, irreducibility of parabolically induced representations of a p-adic reductive group has been given in the recent work [AHHV14].*

Lemma 3.4. *For a character ε of B and an irreducible smooth representation σ of K, the space* $\mathrm{Hom}_G(\mathrm{ind}_K^G\sigma,\ \mathrm{ind}_B^G\,\varepsilon)$ *is at most one-dimensional, and it is non-zero if and only if*

$$\varepsilon_0 = \chi_\sigma^s,$$

where ε_0 is the restriction of ε to H_0.

Proof. Using Iwasawa decomposition $G = BK$, this essentially results from Frobenius reciprocity (See [Abd11, Theorem 4.5.20], [Xu14, Lemm 3.23]). □

By Frobenius reciprocity, Lemma 3.4 describes the irreducible smooth representations σ appearing in a principal series $\mathrm{ind}_B^G\,\varepsilon$.

When the condition of the above Lemma is satisfied, there is a unique $c = c_\varepsilon \in \overline{\mathbf{F}}_p$ such that $* \circ T = c \cdot *$, for any non-zero $*$ in the space $\mathrm{Hom}_G(\mathrm{ind}_K^G\sigma,\ \mathrm{ind}_B^G\,\varepsilon)$.

Proposition 3.5. *The value of c_ε is explicitly known:*

$$c_\varepsilon = \begin{cases} \varepsilon(\alpha) - \bar{\chi}_1(-1), & \textit{if}\ \varepsilon_0 = \varepsilon_0^s = \chi_1 \otimes \chi_2; \\ \varepsilon(\alpha), & \textit{otherwise.} \end{cases}$$

Remark 3.6. *Fix a non-zero $v_0 \in \sigma^{I_{1,K}}$, using Iwasawa decomposition $G = BK$, there is an explicit and well-defined $* = P_{v_0',1}$. We describe the corresponding element $P_{v_0',0}$ in* $\mathrm{Hom}_K(\sigma, \mathrm{ind}_B^G\varepsilon)$*, via Frobenius reciprocity. For $v \in \sigma$, $g = bk \in G$, define $P_{v_0',0}(v)(g) = \varepsilon(b) l_{v_0'}(k \cdot v)$, where l'_{v_0} is the linear functional on $\sigma_{I_{1,K}}$, sending v_0' to 1.*

Proof. (Sketch) To compute the constant c_ε, we only need to evaluate the map $* \circ T = c \cdot *$ at the function $[Id, v_0]$, as it generates the underlying space of the compact induction. We may take $*$ as the map $P_{v_0',1}$ in the above remark. The results then comes from straightforward computation, using the formula of T (Proposition 4). For more details, we refer the readers to [Xu14, Proposition 3.24]. □

Proposition 3.7. (1). *The space* $\mathrm{Hom}_G(\mathrm{ind}_K^G\sigma,\ \chi \circ \det)$ *is at most one-dimensional, and it is non-zero if and only if* $\sigma = \chi \circ \det|_K$.

(2). *The space* $\mathrm{Hom}_G(\mathrm{ind}_K^G\sigma,\ \chi\circ\det\otimes St)$ *is at most one-dimensional, and it is non-zero if and only if* $\sigma=\chi\circ\det\otimes St\mid_K$.

(3). *When the space in* (1) *or* (2) *is non-zero, the eigenvalue for the Hecke operator* T *is* $1-\chi(-1)$.

Proof. (1) is trivial. For (2), it suffices to verify that the K-subrepresentation generated by the St^{I_1} is isomorphic to the finite Steinberg representation st. For (3), we refer the readers to [Abd11, Corollary 4.5.28, Theorem 4.5.29]. ☐

We normalize the operator T in the following way, in view of Proposition 3.5, 3.7.

Definition 3.8. *Let* T_σ *be the following refined Hecke operator*

$$T_\sigma=\begin{cases}T+\bar{\chi}_1(-1), & \textit{if}\ \chi_\sigma=\chi_\sigma^s=\chi_1\otimes\chi_2,\\ T, & \textit{otherwise.}\end{cases}$$

Hence, we may re-write Proposition 3.5 as $*\circ T_\sigma=\varepsilon(\alpha)\cdot *$, for any non-zero $*\in\mathrm{Hom}_G(\mathrm{ind}_K^G\sigma,\ \mathrm{ind}_B^G\,\varepsilon)$. Then, we put:

Definition 3.9. *An irreducible smooth representation* π *of* G *is called supersingular if it is a quotient of* $\mathrm{ind}_K^G\sigma/(T_\sigma)$*, for some irreducible smooth representation* σ *of* K.

Remark 3.10. *In recent literatures ([Her11a], [AHHV14], etc.), Satake isomorphism plays a critical role in the process of defining supersingular representations, and that is also the approach used in the first name author's thesis, which has the advantage to sort out the 'right' Hecke operator, with respect to which supersingular representations are exactly those with eigenvalue zero. However, as the group* G *in this paper is very small, the Hecke operator* T_σ *defined above is normalized from the Hecke operator* T *(2.7) after exhausting the Hecke eigenvalues of all irreducible principal series, which is the original approach in [BL94] and [BL95].*

3.2 Proof of (2) of Theorem 3.1: the unramified case

Let K be a maximal compact open subgroup of G, and $\mathbf{v}_K$ be the unique vertex fixed by K. Without loss of generality, we may assume $K=K_0$ or K_1; as a result $\mathbf{v}_K$ is either $\mathbf{v}_0$ or $\mathbf{v}_1$. Denote by $\triangle(\mathbf{v}_K)$ the set of vertices in the orbit of $\mathbf{v}_K$. We identify the underlying space $\mathscr{I}(K)$ of $\mathrm{ind}_K^G 1$ with the space $C_0(\mathbf{v}_K)$ of all 0-chains which are supported in $\triangle(\mathbf{v}_K)$. Let Deg be the map from $C_0(\mathbf{v}_K)$ to $\overline{\mathbf{F}}_p$: $\mathrm{Deg}(c)=\sum\alpha_v$, for $c=\sum_v\alpha_v\cdot v$, where $\alpha_v\in\overline{\mathbf{F}}_p$.

This map is a surjective G-morphism and trivial on $T(\mathscr{I}(K))$. We denote by $\overline{\mathrm{Deg}}$ the induced map.

Proposition 3.11. (1). *The kernel of* $\overline{\mathrm{Deg}}$ *is isomorphic to the Steinberg representation* St.

(2). *The induced short exact sequence is non-split:*

$$0 \to St \to V_0 \to \mathit{Triv} \to 0,$$

where we write $\mathscr{I}(K)/(T)$ *as* V_0.

We prove (2) at first. we may assume $\mathbf{v}_K = \mathbf{v}_0$ or $\mathbf{v}_1$. We will deal with the case $\mathbf{v}_K = \mathbf{v}_0$ in detail, and the other case follows by modifying the argument in an obvious way.

Suppose the sequence is split. Then, we have a pull-back $\bar{c}$ of $1 \in \overline{\mathbf{F}}_p$, which is G-invariant. Let $c \in \mathscr{I}(K)$ be a representative of $\bar{c}$. Hence, $g\cdot c - c \in T(\mathscr{I}(K))$ for any $g \in G$. Assume the support of c is contained in the ball $B_{2k}(\mathbf{v}_0)$ for some integer $k \geq 0$. Take $g = \alpha^{2k+1}$. For a 0-chain $a \in C_0(\mathbf{v}_K)$, let $\overline{\mathrm{supp}\, a}$ be the set of vertices ($\subseteq \triangle(\mathbf{v}_K)$) of the minimal subtree of $\triangle$ containing $\mathrm{supp}\ a$. We see $\overline{\mathrm{supp}\ (g\cdot c - c)} \subset B_{2k}(\mathbf{v}_0) \cup \alpha^{2k+1}B_{2k}(\mathbf{v}_0) = B_{2k}(\mathbf{v}_0) \cup B_{2k}(\mathbf{v}_{4k+2})$, which we denote by X. Write $g\cdot c - c = T\ b$ for some 0-chain $b \in C_0(\mathbf{v}_0)$. We then claim that $\mathrm{supp}\, b \subset X - \{\mathbf{v}_{2k},\ \mathbf{v}_{2k+2}\}$. Firstly, it is contained in X, from the definition of T and that of the minimal subtree. Secondly, for $\mathbf{v} = \mathbf{v}_{2k}$, or $\mathbf{v}_{2k+2}$, there is some vertex $\mathbf{v}'$ of distance 2 from $\mathbf{v}$, which is not in X. *Moreover*, we can also choose such a $\mathbf{v}'$ which is not a neighbour of $\mathbf{v}_{2k+1}$. Then, if $\mathbf{v}$ is in $\mathrm{supp}\, b$, $\mathbf{v}'$ would lie in $\mathrm{supp}\ (g\cdot c - c)$, a contradiction. Therefore, it is safe to write b as a unique sum $b_1 + b_2$ of two 0-chains, where $\mathrm{supp}\, b_1 \subset X_1 = B_{2k}(\mathbf{v}_0) - \mathbf{v}_{2k}$ and $\mathrm{supp}\, b_2 \subset X_2 = B_{2k}(\mathbf{v}_{4k+2}) - \mathbf{v}_{2k+2}$. As now $d(X_1,\ X_2) \geq 6$, $\mathrm{supp}\ (T\ b_1)$ and $\mathrm{supp}\ (T\ b_2)$ are disjoint. Hence, by comparing the supports, $T\ b_1 = -c$, i.e., $\bar{c} = 0$.

To prove (1), we need some preparation.

Let Λ be a variable, and set $R = \overline{\mathbf{F}}_p[\Lambda, \Lambda^{-1}]$. Define an unramified character $X : E^\times \to R^\times$, by $X(\varpi_E) = \Lambda^{-1}$. We form the character $X \otimes 1$ of T by: $X \otimes 1(t) = X(x)$, where t is the matrix:

$$\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & \bar{x}^{-1} \end{pmatrix}.$$

Then we view $X \otimes 1$ as a character of B which is trivial on the subgroup N, and define the principal series. The choice of the character implies there

is non-trivial function f_0 in the principal series which is K-invariant, i.e., by writing an element g as bk, $f_0(bk) = X \otimes 1(b)$, where $b \in B$, $k \in K$. Especially, $f_0(\beta_K) = 1$.

Consider the map from N to $E^\times \times E^\times : n(x, y) \mapsto (x, y)$, and denote by N^* (resp, N_k^*) the image of the group N (resp, N_k), and it inherits a group structure from that of N (resp, N_k). Define $\mathscr{J}(X \otimes 1)$ as the space of locally constant functions φ from N^* to R which satisfy $\varphi((x, y)) = c \cdot \Lambda^{\mathrm{val}(y) - n_K}$ for some constant $c = \mathrm{const}(\varphi) \in R$, when y is large enough. Therefore by the above, the map i which maps a f to $i(f)$, where $i(f)((x, y)) = f(\beta_K n(x, y))$, is an isomorphism from $\mathscr{I}(X \otimes 1)$ to $\mathscr{J}(X \otimes 1)$. The inverse j of i explicitly: for a function φ in $\mathscr{J}(X \otimes 1)$

$$j(\varphi)(g) = \begin{cases} \mathrm{const}(\varphi) X \otimes 1(b), & \text{when } g = b \in B, \\ X \otimes 1(b)\varphi((x, y)), & \text{when } g = b\beta_K n(x, y). \end{cases} \tag{5}$$

The space $\mathscr{J}(X \otimes 1)$ then inherits a structure of G-module. We record the function φ_0, which corresponds to f_0 via the map i:

$$\varphi_0((x, y)) = \begin{cases} \Lambda^{\mathrm{val}(y) - n_K}, & \text{if } \mathrm{val}(y) \leq n_K, \\ 1, & \text{if } \mathrm{val}(y) \geq n_K. \end{cases}$$

Let $\mathscr{S}(N^*, R)$ be the space of locally constant functions on N^*, which take values in R and have compact support. By definition, $\mathscr{S}(N^*, R)$ is a subspace of $\mathscr{J}(X \otimes 1)$, which has a set $\{1_{N_k^* \cdot (x,y);\ k \in \mathbb{Z},\ (x,y) \in N^*}\}$ of characteristic functions as generators, and there exists a direct sum decomposition: $\mathscr{J}(X \otimes 1) = \mathscr{S}(N^*, R) \oplus R\varphi_0$.

Lemma 3.12. *For* $\varphi \in \mathscr{J}(X \otimes 1)$,

(1). $n(x', y')\varphi((x, y)) = \varphi((x + x', y + y' - x\bar{x}'))$.
(2). $\alpha\varphi((x, y)) = \Lambda^{-1}\varphi((\varpi_E x, \varpi_E^2 y))$.
(3). $\alpha^{-1}\varphi((x, y)) = \Lambda\varphi((\varpi_E^{-1} x, \varpi_E^{-2} y))$.

Proposition 3.13. $\varphi_0 \mid T = (\Lambda - 1)\varphi_0$

Proof. This is a special case of Proposition 3.5. ∎

As f_0 is K-invariant, φ_0 is also K-invariant. This K-invariant function gives rise to a G-morphism $\phi_{\varphi_0}^K$ from $\mathrm{ind}_K^G 1$ to $\mathrm{ind}_B^G X \otimes 1$ which corresponds to φ_0 by Frobenius reciprocity, i.e., $\phi_{\varphi_0}^K(1_K) = \varphi_0$. This morphism extends to an R-linear morphism from the underlying space $\mathscr{V}$ of $\mathrm{ind}_K^G 1 \otimes_{\overline{\mathbf{F}}_p} R$ to $\mathscr{J}(X \otimes 1)$, which we also denote by $\phi_{\varphi_0}^K$.

We are interested in the properties of $\phi_{\varphi_0}^K$.

Proposition 3.14. $\sum_{u \in N_{n_K-1}/N_{n_K}} u \cdot \varphi_0 = (1-\Lambda^{-1}) 1_{N^*_{n_K-1}}$

Proof. This comes from explicit computations, using equality (1). ☐

Theorem 3.15. (1). *The image of $\mathscr{V}$ under $\phi^K_{\varphi_0}$ is $(1-\Lambda^{-1})\mathscr{S}(N^*, R) \oplus R\varphi_0$.*

(2). *The kernel of $\phi^K_{\varphi_0}$ is $(T - \Lambda + 1)\mathscr{V}$.*

Proof. We deal with (1) first. From (3) of Lemma 3.12, we get $\alpha^{-n} 1_{N^*_{n_K-1}} = \Lambda^n 1_{N^*_{2n+n_K-1}}$. Then for any integer n, $(1-\Lambda^{-1}) 1_{N^*_{2n+n_K-1}}$ is in the image of $\phi^K_{\varphi_0}$ by Proposition 3.14.

By (1) of Lemma 3.12, $n(x,y) 1_{N^*_l} = 1_{N^*_l(x,y)^{-1}}$. This shows that, for any $(x,y) \in N^*$ and any integer n, $(1-\Lambda^{-1}) 1_{N^*_{2n+n_K-1}\cdot(x,y)}$ lies in the image of $\phi^K_{\varphi_0}$. Furthermore, we have

$$1_{N^*_{2n+n_K-1}} = \sum_{u \in N_{2n+n_K-1}/N_{2n+n_K}} u \cdot 1_{N^*_{2n+n_K}}, \tag{6}$$

and using (1) of Lemma 3.12 again, we see that, for any $(x,y) \in N^*$ and any integer n, $(1-\Lambda^{-1}) 1_{N^*_{2n+n_K-1}\cdot(x,y)}$ lies in the image of $\phi^K_{\varphi_0}$. We have proved $(1-\Lambda^{-1})\mathscr{S}(N^*, R) \oplus R\varphi_0$ is contained in the image of $\phi^K_{\varphi_0}$.

Now for a vertex $\mathbf{v} \in \triangle(\mathbf{v}_K)$, there is a unique path from $\mathbf{v}$ to $\mathbf{v}_K$; For simplicity of notations, we write $\mathbf{v}_K$ as $\mathbf{v}_d$, for $d = 0, 1$. We could express $\mathbf{v}$ as $\sum_l t_l(\mathbf{v}_l - \Lambda^{-1} a^2(\mathbf{v}_l)) + t_0 \mathbf{v}_d$, for $t \in R$. This expression of $\mathbf{v}$ changes into $\sum_l t_l g_l(\mathbf{v}_d - \Lambda^{-1}\mathbf{v}_{d+2}) + t_0\mathbf{v}_d$ for some g_l in G. Then $\phi^{K_0}_{\varphi_0}(\mathbf{v}) = \sum_l t_l g_l(\varphi_0 - \Lambda^{-1}\alpha\varphi_0) + t_0\varphi_0$.

By the definition of φ_0 and (2) of Lemma 3.12. we compute $\varphi_0 - \Lambda^{-1}\alpha\varphi_0 = (1-\Lambda^{-1})(\Lambda^{-1} 1_{N^*_{n_K-1}} + 1_{N^*_{n_K}})$. We also note that $g\varphi$ is in $\mathscr{S}(N^*, R) \oplus R\varphi_0$, for any $g \in G$ and $\varphi \in \mathscr{S}(N^*, R)$. This shows that $\phi^K_{\varphi_0}(\mathbf{v})$ is in the space $(1-\Lambda^{-1})\mathscr{S}(N^*, R) \oplus R\varphi_0$. This finishes our argument.

We now prove (2). Firstly, by Proposition 3.13, we have $\phi^K_{\varphi_0}((T - \Lambda + 1)(1_K)) = 0$. As the G-translates of 1_K generate $\mathscr{V}$, we conclude that $\phi^K_{\varphi_0}$ vanishes on $(T - \Lambda + 1)\mathscr{V}$.

Given $c \in \mathscr{V}$ such that $\phi^K_{\varphi_0}(c) = 0$, we write c as $\sum_{\mathbf{v}\in S} t_{\mathbf{v}} \cdot \mathbf{v}$, where S is a finite set of $\triangle(\mathbf{v}_K)$. So we can find a vertex $\mathbf{v}_r \in \triangle(\mathbf{v}_K)$ in the standard apartment such that $\mathbf{v}_r \in \cap_{\mathbf{v}\in S} \overline{\mathbf{v}\infty}$, i.e., all the vertices in S are under $\mathbf{v}_r$. We put $s = \min_{\mathbf{v}\in S} h(\mathbf{v})$. Note that $s \equiv r (\mathrm{mod}\ 2)$. Then if we allow some $t_{\mathbf{v}}$ to be zero, we can assume S to be the finite subset of $\triangle(\mathbf{v}_K)$ consisting of all the vertices under $\mathbf{v}_r$ and with height greater than or equal to s.

There is an equality:

$$\mathbf{v} = \Lambda^{-1}a^2(\mathbf{v}) - \Lambda^{-1}(a^2(\mathbf{v}) - \Lambda\mathbf{v}).$$

Replacing each $\mathbf{v} \in S$ in the expression of c by the right side of the equality above (doing this from the vertices of least height and moving up), we get:

$$c = P \cdot \mathbf{v}_r + \sum_{\mathbf{v}\in S, \mathbf{v}\neq\mathbf{v}_r} P_{\mathbf{v}} \cdot (a^2(\mathbf{v}) - \Lambda\mathbf{v}), \tag{7}$$

where P and $P_{\mathbf{v}}$ are polynomials in Λ, Λ^{-1}.

Recall that for $\mathbf{v} \in \triangle(\mathbf{v}_K)$, $T\mathbf{v} = \sum_{d(\mathbf{v}',\mathbf{v})=2} \mathbf{v}'$ ((4)). Then we get that for such a $\mathbf{v}$,

$$T\mathbf{v} + \mathbf{v} = a^2(\mathbf{v}) + \sum_{a(\mathbf{v}')=a(\mathbf{v})} \mathbf{v}' + \sum_{a^2(\mathbf{v}')=\mathbf{v}} \mathbf{v}'.$$

Note that the numbers of terms appearing in the second and the third sum above are respectively q (or q^3) and q^4. Then a rearrangement gives

$$T\mathbf{v}+\mathbf{v}-\Lambda\mathbf{v} = a^2(\mathbf{v})-\Lambda\mathbf{v}+\Lambda^{-1}\sum_{a(\mathbf{v}')=a(\mathbf{v})}(\Lambda\mathbf{v}'-a^2(\mathbf{v}'))+\Lambda^{-1}\sum_{a^2(\mathbf{v}')=\mathbf{v}}(\Lambda\mathbf{v}'-a^2(\mathbf{v}')). \tag{8}$$

Hence,

$$a^2(\mathbf{v}) - \Lambda\mathbf{v} \equiv \Lambda^{-1}\sum_{a(\mathbf{v}')=a(\mathbf{v})}(a^2(\mathbf{v}') - \Lambda\mathbf{v}') + \Lambda^{-1}\sum_{a^2(\mathbf{v}')=\mathbf{v}}(a^2(\mathbf{v}') - \Lambda\mathbf{v}'). \tag{9}$$

where the congruences appearing above and below are all $\bmod (T-\Lambda+1)\mathscr{V}$.

Note that in the first sum on the right side of (9), $\mathbf{v}'$ goes through all the vertices under and adjacent to $a(\mathbf{v})$, which particularly means that these $\mathbf{v}'$ are of the same height. So the height is not reduced if we insert (9) directly into the expression (7) of c that we got in step one.

Now write $a(\mathbf{v})$ as $\mathbf{u}$. Viewing $\mathbf{u}$ as fixed, we sum (9) over the vertices $\mathbf{v}''$ which are under and adjacent to $\mathbf{u}$. Then the first sum on the right of (9) disappears as it becomes a constant and is counted q (or q^3) times. We get

$$\begin{aligned}\sum_{a(\mathbf{v}'')=\mathbf{u}}(a^2(\mathbf{v}'') - \Lambda\mathbf{v}'') &\equiv \sum_{a(\mathbf{v}'')=\mathbf{u}}\Lambda^{-1}\sum_{a^2(\mathbf{v}')=\mathbf{v}''}(a^2(\mathbf{v}') - \Lambda\mathbf{v}')\\ &\equiv \Lambda^{-1}\sum_{a^3(\mathbf{v}')=\mathbf{u}}(a^2(\mathbf{v}') - \Lambda\mathbf{v}')\end{aligned}$$

Then by inserting the above into the right side of (9), we finally obtain

$$a^2(\mathbf{v}) - \Lambda\mathbf{v} \equiv \Lambda^{-2}\sum_{a^3(\mathbf{v}')=a(\mathbf{v})}(a^2(\mathbf{v}') - \Lambda\mathbf{v}') + \Lambda^{-1}\sum_{a^2(\mathbf{v}')=\mathbf{v}}(a^2(\mathbf{v}') - \Lambda\mathbf{v}'). \tag{10}$$

We also note that the right side of (10) can be written as

$$\sum_{a^3(\mathbf{v}')=a(\mathbf{v})} Q_{\mathbf{v}'} \cdot (a^2(\mathbf{v}') - \Lambda \mathbf{v}'),$$

where $Q_{\mathbf{v}'}$ is some polynomial in Λ and Λ^{-1} (depending on $\mathbf{v}'$).

Using (10) for all $\mathbf{v} \in S$ except $\mathbf{v}_r$ (starting from the top and going down, in (7)), we get

$$c \equiv P \cdot \mathbf{v}_r + \sum_{\mathbf{v}\in S,\ h(\mathbf{v})=s} P'_{\mathbf{v}} \cdot (a^2(\mathbf{v}) - \Lambda \mathbf{v}).$$

Our assumption is $\phi^K_{\varphi_0}(c) = 0$. Then by Proposition 3.13, the congruence above gives

$$0 = P \cdot \phi^K_{\varphi_0}(\mathbf{v}_r) + \sum_{\mathbf{v}\in S,\ h(\mathbf{v})=s} P'_{\mathbf{v}} \cdot \phi^K_{\varphi_0}(a^2(\mathbf{v}) - \Lambda \mathbf{v}). \tag{11}$$

We need to compute the right side of the equation more explicitly. Firstly, as $r = 2r_1$ or $2r_1 + 1$,

$$\phi^K_{\varphi_0}(\mathbf{v}_r) = \phi^K_{\varphi_0}(\alpha^{r_1}\mathbf{v}_K) = \alpha^{r_1}\varphi_0.$$

Secondly, from the proof of Lemma 2.3, we know that

$$\{v \in S | h(\mathbf{v}) = s\} = (N_{-r}/N_{-s})\mathbf{v}_s.$$

Then given $u = n(x, y) \in N_{-r}$, i.e., $y \in \mathfrak{p}_E^{-r}$, from Lemma 2.5 and Lemma 3.12 we see that (writing $s = 2s_1$ or $2s_1 + 1$)

$$\begin{aligned}
\phi^K_{\varphi_0}(a^2(u\mathbf{v}_s) - \Lambda u\mathbf{v}_s) &= \phi^K_{\varphi_0}(u\mathbf{v}_{s+2} - \Lambda u\mathbf{v}_s)\\
&= \phi^K_{\varphi_0}(u\alpha^{s_1}(\alpha\mathbf{v}_K - \Lambda\mathbf{v}_K))\\
&= u\alpha^{s_1}(\alpha\varphi_0 - \Lambda\varphi_0)\\
&= (\Lambda^{-1} - 1)\Lambda^{-s_1}(1_{N^*_{n_K-2s_1-1}\cdot(x,y)^{-1}} + \Lambda \cdot 1_{N^*_{n_K-2s_1}\cdot(x,y)^{-1}}),
\end{aligned}$$

from which we can see that the supports of the functions $\phi^K_{\varphi_0}(a^2(u\mathbf{v}_s) - \Lambda u\mathbf{v}_s)$ may intersect as $u = n(x, y)$ goes through N_{-r}/N_{-s}.

One observes from these computations that $\phi^K_{\varphi_0}(\mathbf{v}_r)$ is of non-compact support, but all the other $\phi^K_{\varphi_0}(a^2(\mathbf{v}) - \Lambda\mathbf{v})$ have compact support. Therefore we can conclude that $P \equiv 0$. Then by substituting the display above, (11) turns into

$$\sum_{u=n(x,y)\in N_{-r}/N_{-s}} P'_{n(x,y)} \cdot (1_{N^*_{n_K-2s_1-1}\cdot(-x,\bar{y})} + \Lambda \cdot 1_{N^*_{n_K-2s_1}\cdot(-x,\bar{y})}) = 0,$$

where we write $P'_{n(x,y)}$ for $P'_{\mathbf{v}}$, for $\mathbf{v} = n(x,y)\mathbf{v}_s$. When decomposing $N^*_{n_K-2s_1-1}$ into $\bigcup_{u_i} N^*_{n_K-2s_1} \cdot u_i$, and re-writing the sum over the left cosets, the above equation turns into

$$\sum_{\substack{u' \in N_{-s} \backslash N_{-r}, \\ u'=n(x,y)}} P'_{n(-x,\bar{y})} \cdot \left((1+\Lambda) 1_{N^*_{n_K-2s_1} \cdot (x,y)} + \sum_{u_i \neq 1} 1_{N^*_{n_K-2s_1} \cdot u_i \cdot (x,y)} \right) = 0. \tag{12}$$

Note that here $n_K - 2s_1 = -s$.

For simplicity, we rewrite $P'_{n(-x,\bar{y})}$ above as $P''_{n(x,y)}$. To deal with (12), we note first that $u \cdot n(x',y')$ goes through $N_{-s} \backslash N_{-r}$ when $n(x',y')$ and u go through $N_{-s-1} \backslash N_{-r}$ and $N_{-s} \backslash N_{-s-1}$ respectively.

Then another observation we need is that: for a given $u = n(x,y) \in N_{-s} \backslash N_{-r}$, when u_i goes through $N_{-s} \backslash N_{-s-1}$, $N^*_{-s} \cdot u_i \cdot (x,y)$ also goes through $N^*_{-s} \cdot (x,y) \backslash N^*_{-s-1} \cdot (x,y)$.

With these in mind, we can see that for a fixed $(x',y') \in N^*_{-s-1} \backslash N^*_{-r}$, the coefficient of a characteristic function $1_{N^*_{-s} \cdot u_i \cdot (x',y')}$ (appearing in (12)) is $(1+\Lambda) P''_i + \sum_{j \neq i} P''_j$, where P''_i (relative to (x',y')) is short for $P''_{u_i \cdot n(x',y')}$. Therefore we can rewrite (12) as:

$$\sum_{\substack{u' \in N_{-s-1} \backslash N_{-r}, \\ u'=n(x',y')}} \sum_{u_i \in N_{-s} \backslash N_{-s-1}} ((1+\Lambda) P''_i + \sum_{j \neq i} P''_j) \cdot 1_{N^*_{-s} \cdot u_i \cdot (x',y')} = 0. \tag{13}$$

Now from (13) we can conclude that for a fixed $(x',y') \in N^*_{-s-1} \backslash N^*_{-r}$,

$$(1+\Lambda) P''_i + \sum_{j \neq i} P''_j = 0, \ \ u_i \in N_{-s} \backslash N_{-s-1}. \tag{14}$$

It remains to solve out $\{P''_i\}_i$ from the equations (14). In fact, by adding together all the equations in (14), we get $\sum_{u_i} (1 + \Lambda + q^* - 1) P''_i = 0$, which is just

$$\sum_{u_i} P''_i = 0. \tag{15}$$

Subtracting (15) from every equation in (14), we obtain that all the P''_i are 0.

Changing back the notations, we have indeed shown that $P''_{n(x,y)}$ are all 0, for $n(x,y) \in N_{-s} \backslash N_{-r}$, i.e., $P'_{n(x,y)}$ are all 0, for $n(x,y) \in N_{-r}/N_{-s}$. We have finally proved $c \equiv 0$, i.e., $c \in (T - \Lambda + 1)\mathscr{V}$. We are done. $\square$

Now we are ready to prove (1) of Proposition 3.11.

Let $\mathscr{V}'$ be the underlying space of the representation $\mathrm{ind}_K^G 1 \otimes_{\mathcal{H}_K} R$. Then we have an isomorphism

$$\mathscr{V}'/T\mathscr{V}' \cong V_0 = \mathscr{I}(K)/T\mathscr{I}(K).$$

Hence, we are given a degree map:

$$\overline{\mathrm{Deg}}:\ V_0 = \mathscr{V}'/T\mathscr{V}' \to \overline{\mathbf{F}}_p. \tag{16}$$

We now apply [BL95, Lemma 31] to our situation: $D = R$, $P = (\Lambda - 1)$, $S =$ the group algebra $\overline{\mathbf{F}}_p[G]$, $Y' = \mathscr{V}'$, $Y = \mathscr{J}(X \otimes 1)$, hence we view both Y and Y' as a $(S,\ R)$-bi-module. Then, we use Theorem 3.15:

We indeed have $\mathscr{V}'/T\mathscr{V}' = \mathscr{V}'/(\Lambda - 1)\mathscr{V}'$, from Proposition 3.13 . On the other hand, $(\Lambda - 1)\mathscr{J}(X \otimes 1)$ is contained in the image of $\mathscr{V}'$ under the injection $\phi^K_{\varphi_0}$ by (1) of Theorem 3.15. So the condition of [BL95, Lemma 31] is satisfied. As an $\overline{\mathbf{F}}_p[G]$-module, $\mathscr{V}'/T\mathscr{V}'$ and $\mathscr{J}(X \otimes 1)/(\Lambda - 1)\mathscr{J}(X \otimes 1)$ have the same length and the same Jordan-Hölder factors. However, $\mathscr{J}(X \otimes 1)/(\Lambda - 1)\mathscr{J}(X \otimes 1)$ is just $\mathscr{J}(1 \otimes 1)$, i.e., the space of the representation $\mathrm{ind}^G_B 1$, which is of length 2 with two Jordan-Hölder factors: Triv, St. Hence, the Kernel of $\overline{\mathrm{Deg}}$, as an $\overline{\mathbf{F}}_p[G]$-module, must be irreducible and isomorphic to St.

For a non-zero $\lambda \in \overline{\mathbf{F}}_p$, let χ_λ be the unramified character of $E^\times$, which takes value λ^{-1} at ϖ_E.

Theorem 3.16. *Let (π, V) be an irreducible smooth representation of G such that $V^K \neq 0$. Then,*

(1). There exist a vector $v \neq 0$ in V^K which is an eigenvector for $\mathcal{H}_K$.

(2). Let v be an eigenvector in (1), and denote by λ the corresponding eigenvalue, i.e., $v \mid T = \lambda v$. Suppose $\lambda \neq -1$. Then,

(a). If $\lambda \neq 0$, then $\mathrm{dim} V^K = 1$ and $(\pi, V) \cong \mathrm{ind}^G_B \chi_{\lambda+1} \otimes 1$;

(b). If $\lambda = 0$, then $\mathrm{dim} V = 1$, and $(\pi, V) \cong$ Triv.

Proof. (1). This is a special case of (1) of Theorem 3.1, and its proof is in the appendix A.

(2). For v as in (1), denote by λ the corresponding eigenvalue, i.e., $v \mid T = \lambda v$. Assume that $\lambda \neq -1$. By the definition of the right action, ϕ^K_v is trivial on $(T - \lambda)\mathscr{I}(K)$. So (π, V) is equivalent to an irreducible quotient of

$$\mathrm{ind}^G_K 1/(T - \lambda)\mathrm{ind}^G_K 1$$

via the map ϕ^K_v.

For (b), where $\lambda = 0$. By Proposition 3.11, $\mathrm{ind}^G_K 1/(T)\mathrm{ind}^G_K 1$ contains the Steinberg representation St, with quotient Triv. As St is the unique subrepresentation of $\mathrm{ind}^G_K 1/(T)\mathrm{ind}^G_K 1$ (from (2) of Proposition 3.11), we conclude that $\pi \cong$ Triv.

For (a), where $\lambda \neq 0$. As $\lambda+1 \neq 0$, we can form the principal series $\mathrm{ind}_B^G(\chi_{\lambda+1} \otimes 1)$ with underlying space $\mathscr{I}(\chi_{\lambda+1} \otimes 1)$, and it is irreducible as $\lambda+1 \neq 1$. The K-invariant function φ_0 in $\mathscr{I}(\chi_{\lambda+1} \otimes 1)$ gives rise to a G-morphism $\phi_{\varphi_0}^K$ from $\mathrm{ind}_K^G 1$ to $\mathrm{ind}_B^G(\chi_{\lambda+1} \otimes 1)$. From Proposition 3.13, we see $\varphi_0 \mid T = \lambda\varphi_0$. Hence $\phi_{\varphi_0}^K$ is trivial on $(T-\lambda)\mathrm{ind}_K^G 1$ and we get an induced morphism:

$$\phi_{\varphi_0}^K : \mathrm{ind}_K^G 1/(T-\lambda)\mathrm{ind}_K^G 1 \to \mathrm{ind}_B^G(\chi_{\lambda+1} \otimes 1). \tag{17}$$

Now the right side of the above is irreducible. From the conditions that $\lambda+1 \neq 1$ and $\lambda+1 \neq 0$, the same argument (changing Λ into $\lambda+1$) in proving (2) of Theorem 3.15 will imply that the $\phi_{\varphi_0}^K$ above is injective. As it is non-zero, $\phi_{\varphi_0}^K$ is an isomorphism. We conclude (π, V) is equivalent to $\mathrm{ind}_B^G(\chi_{\lambda+1} \otimes 1)$. $\square$

3.3 Injectivity from $\mathrm{ind}_K^G\sigma/(T-\lambda)$ to $\mathrm{ind}_B^G\varepsilon$: $\lambda \neq 0$

Based on the formula of T ((4)), we generalize the 2-antecedent on the tree partially in the following way: fix a non-zero $v_0 \in \sigma^{I_{1,K}}$, and put $v_0' = \beta_K v_0$.

Definition 3.17. $A[u_0\alpha^k, v] = \begin{cases} [u_0\alpha^{k+1}, \sigma(\beta_K)v], & \text{if } \dim\sigma = 1, \\ [u_0\alpha^{k+1}, \sigma(\beta_K)\cdot j_\sigma \cdot \sigma(\beta_K)\cdot v)], & \text{otherwise.} \end{cases}$

Recall from Proposition 3.5 that, $P_{v_0',1} \circ T = c_\varepsilon \cdot P_{v_0',1}$. The main result of this section is the following:

Proposition 3.18. *When* $\dim\sigma > 1$ *and* $c_\varepsilon \neq 0$, $P_{v_0',1}$ *is injective.*

Proof. Denote c_ε by λ. Under the assumption $\lambda \neq 0$, we start from the formula (4) of T and rephrase it as follows:

Lemma 3.19. $T[u_0\alpha^k, v] - \lambda[u_0\alpha^k, v]$

$$= A[u_0\alpha^k, v] - \lambda[u_0\alpha^k, v] + \lambda^{-1}\sum_u (\lambda \cdot \mathfrak{f}_{0,u} - A \cdot \mathfrak{f}_{0,u})$$

$$+ \lambda^{-1}\sum_u (\lambda \cdot \mathfrak{f}_{1,u} - A \cdot \mathfrak{f}_{1,u})$$

Proof. The formula in the statement is essentially a re-written of the Hecke operator T, and here we only record how those functions on the right hand side look like:

$$f_{0,u} = [u\alpha^k, c_{0,u}v_0], \text{ for some } u \in N_{-2k-1}$$
$$f_{1,u} = [u\alpha^{k-1}, c_{1,u}v_0], \text{ for some } u \in N_{-2k}.$$

Here, $c_{1,u}$ and $c_{2,u}$ are some constants related to $u \in N$ and $v \in \sigma$. Note that $A[u\alpha^k, v_0] = 0$, under the assumption $\dim \sigma > 1$ (Remark 2.6). $\square$

Using the above Lemma repeatedly, we get the following Corollary (compare with (8) in the proof of Theorem 3.15)

Corollary 3.20. *For an $u_0 \in N$, $k \in \mathbb{Z}$, and $v \in W$, we have*

$$A[u_0\alpha^k,\ v] - \ \lambda[u_0\alpha^k,\ v] \equiv \textstyle\sum_j\ f_j(A[u_j\alpha^{k-1},\ v_j] - \ \lambda[u_j\alpha^{k-1},\ v_j])$$

for some $f_j \in \overline{\mathbf{F}}_p$ and some vectors $v_j \in W$. The elements u_j are all in N, satisfying that the vertices $u_j\alpha^{k-1}\mathbf{v}_K$ are distinct from each other. Here the congruence is taken modulo $(T-\lambda)$.

Based on last Corollary, we follow the process of Theorem 3.15 to prove $P_{v_0',1}$ is indeed injective.

Let $c \in S(G,\sigma)$ such that $P_{v_0',1}(c) = 0$. We write c as $\sum_{j\in S}\ [u_j\alpha^j, v_j]$, where $u_j \in N$, $v_j \in W$. Let s be $\min_{j\in S}\{h(u_j\alpha^j\mathbf{v}_K)\}$, and assume all the vertices $u_j\alpha^j\mathbf{v}_K$ are under $\mathbf{v}_{r-2}$. Also, by setting some v_j to be 0, we may enlarge S so that the vertices $u_j\alpha^j\mathbf{v}_K$ go through all the vertices strictly under $\mathbf{v}_r$ and with height at least s. Write $r = 2r_1$ or $2r_1+1$, similarly write $s = 2s_1$ or $2s_1+1$. Using the following identity

$$[u\alpha^k, v] = \lambda^{-1}A[u\alpha^k, v] - \lambda^{-1}(A[u\alpha^k, v] - \lambda[u\alpha^k, v]),$$

we rewrite c as:

$$c = P\cdot[u\alpha^{r_1}, v_0'] + \sum_{s_1<j<r_1} P_j\cdot(A[u_j\alpha^j, v_j] - \lambda[u_j\alpha^j, v_j]), \tag{18}$$

where P, P_j are some constants in $\overline{\mathbf{F}}_p$.

Combining the above equation with Corollary 3.20, we obtain

$$c \equiv P\cdot[u\alpha^{r_1}, v_0'] + \textstyle\sum_{u\in N_{-r}/N_{-s}}\ P_u\cdot(A[u\alpha^{s_1}, v_0] - \lambda[u\alpha^{s_1}, v_0]).$$

Recall we are in the case of $\dim\sigma > 1$, the constant $\lambda_{\beta,\sigma}$ vanishes. By definition of antecedent, $A[u\alpha^{s_1}, v_0] = 0$ for all u.

The function $\mathfrak{f}_0$, which is $P_{v_0',1}[Id, v_0]$, is 0 at Id and 1 at β_K. In our former notation, we would like to use $1_{N^*_{n_K}}$.

We compute first $P_{v_0',1}([u\alpha^{r_1}, v_0']) = u\alpha^{r_1}\beta_K\mathfrak{f}_0 = u\alpha^{r_1}\beta_K 1_{N^*_{n_K}}$, which has non-compact support by pulling-back. Secondly, we compute $P_{v_0',1}([u\alpha^{s_1}, v_0]) = u\alpha^{s_1}\mathfrak{f}_0 = u\alpha^{s_1}1_{N^*_{n_K}} = \varepsilon(\alpha)^{-s}1_{N^*_{n_K-2s_1}\cdot(-x,\bar{y})}$ for $u = n(x,y) \in N_{-r}/N_{-s}$ which is compactly supported. Note that $n_K - 2s_1 = -s$. Hence, we conclude that $P = 0$. For the remaining terms, their supports $N^*_{-s}\cdot(-x,\bar{y})$ are disjoint when $u = n(x,y)$ goes through N_{-r}/N_{-s}. We then conclude all the

P_u are 0. We therefore have shown $c \in (T-\lambda)$. In all, the injectivity of $P_{v_0',1}$ is shown. $\square$

3.4 Twisting $\mathrm{ind}_K^G \sigma/(T_\sigma - \lambda)$ by characters

Let σ be an irreducible smooth representation of K, λ be a scalar in $\overline{\mathbf{F}}_p$. In this part, we record a simple fact describing how $\mathrm{ind}_K^G \sigma/(T_\sigma - \lambda)$ is changed when twisted by a character of G, under the hypothesis that $\lambda \neq 0$. Before stating the result, we recall a little more notation.

Write $\chi_\sigma = \chi_{1,\sigma} \otimes \chi_{2,\sigma}$, the character of I acting on σ^{I_1}. Let σ_η be a twist of σ, i.e., $\sigma_\eta = \eta \circ \det \otimes \sigma$, by some character η of k_E^1. It is clear $\chi_{\sigma_\eta} = \chi_\sigma \cdot (\eta \circ \det)$. We view η as a character of E^1 and assume the character $\eta \circ \det$ of K extends to a character $\eta \circ \det$ of G. A character of E^1 can be viewed as a character of k_E^1; for a character η of E^1, the restriction of the character $\eta \circ \det$ of G to K is just $\eta \circ \det$.

Lemma 3.21. *When $\lambda \neq 0$, there is an isomorphism of G-representations:*

$$\mathrm{ind}_K^G \sigma_\eta/(T_{\sigma_\eta} - \lambda) \cong \eta \circ \det \otimes (\mathrm{ind}_K^G \sigma/(T_\sigma - \lambda)).$$

3.5 Proof of (b) of (2) of Theorem 3.1

Theorem 3.22. *We have an isomorphism of G-representations:*

$$\mathrm{ind}_K^G st/(T) \cong \mathrm{ind}_B^G 1$$

Proof. The image of $P_{v_0',1}$ is generated by the function $P_{v_0',1}([Id, v_0]) = \mathfrak{f}_0$. In this situation, $\mathfrak{f}_0$ is not fixed by K, and $P_{v_0',1}$ is surjective, as $\mathrm{ind}_B^G 1$ has trivial character as the unique proper subrepresentation. By Proposition 3.5, $P_{v_0',1}(T[Id, v_0]) = 0$. It suffices to prove the induced map $P_{v_0',1}$ is injective. However, it seems the strategy used in section 3.3 does not work here. But one may verify that $P_{v_0',1}$ is still surjective when restricted to the subspace of I_1-invariants.

We choose a proper character η of k_E^1, so that $\eta(-1) \neq 1$. Let $\sigma_\eta = \eta \circ \det \otimes St$. The I_1-invariants of σ_1 are generated by v_0, on which I acts as character χ_{σ_η}. Hence, we may use the same notation $P_{v_0',1}$ as the non-zero G-morphism in $\mathrm{Hom}_G(\mathrm{ind}_K^G \sigma_\eta, \mathrm{ind}_B^G\, \eta \circ \det)$. As the trivial character case above, $P_{v_0',1}$ is surjective. By Proposition 3.5, $P_{v_0',1}$ factors through the quotient $\mathrm{ind}_K^G \sigma_\eta/(T - (1-\eta(-1)))$. As $1 - \eta(-1)) \neq 0$ and $\dim \sigma_\eta > 1$, $P_{v_0',1}$ is injective, by Proposition 3.18. We are done, by applying Lemma 3.21. $\square$

We need the following analogue of Proposition 3.11.

Proposition 3.23. *We have the following non-split short exact sequence:*

$$0 \to \mathit{Triv} \to \mathrm{ind}_K^G st/(T) \to St \to 0. \tag{19}$$

Proof. It is implied by Proposition 3.22, and that $\mathrm{ind}_B^G\, 1$ is the non-split extension of St by trivial representation ([Abd11, Theorem 4.1.3]). $\square$

We proceed to complete the proof of (b) of (2) of Theorem 3.1. In this case χ_σ factors through the determinant, i.e., $\chi_\sigma = \eta\circ\det$ for some character η of k_E^1, and $\lambda = 1 - \bar{\chi}_1(-1)$. From the theory of Carter-Lusztig ([KX15, (i) of Lemma 5.8]), $\sigma \cong \eta\circ\det$ or $\sigma \cong \eta\circ\det\otimes St$. In the first case $\mathrm{ind}_K^G\sigma \cong \eta\circ\det\otimes\mathrm{ind}_K^G 1$, and in the second case, $\mathrm{ind}_K^G\sigma \cong \eta\circ\det\otimes\mathrm{ind}_K^G St$. However, from Proposition 3.11 and Proposition 3.23, we conclude that $\eta\circ\det$ (resp. $\eta\circ\det\otimes St$) is the unique quotient of $\mathrm{ind}_K^G\eta\circ\det/(T-(1-\eta(-1)))$ (resp. $\mathrm{ind}_K^G\eta\circ\det\otimes St/(T-(1-\eta(-1)))$. Hence, we are done, by applying Lemma 3.21.

3.6 Proof of (a) of (2) of Theorem 3.1

We finish in this part to prove (a) of (2) of Theorem 3.1.

Before going into details, we recall a little more about the situation of (a). Under the assumption of (a), the principal series $\mathrm{ind}_B^G\varepsilon$ is irreducible and there is a non-zero G-surjective morphism from $\mathrm{ind}_K^G\sigma$ to $\mathrm{ind}_B^G\varepsilon$, which factors through $\mathrm{ind}_K^G\sigma/(T-\lambda)$. We will prove that $\mathrm{ind}_K^G\sigma/(T-\lambda)$ is irreducible, which completes the argument that π is isomorphic to $\mathrm{ind}_B^G\varepsilon$.

For (a), we separate it into two cases:

Case 1 : σ is a character and $\lambda \neq 1 - \bar{\chi}_{1,\sigma}(-1)$.

We repeat that $\mathrm{ind}_B^G\varepsilon$ is irreducible and is a quotient of $\mathrm{ind}_K^G\sigma/(T-\lambda)$. We reduce it to the unramified case which is already known. Write $\sigma = \eta\circ\det$ for a character η of k_E^1. Consider the principal series $\mathrm{ind}_B^G\varepsilon_1$, where $\varepsilon_1 \mid_{H_0} = (\eta^{-1}\circ\det)\cdot\varepsilon\mid_{H_0}$, $\varepsilon_1(\alpha) = \varepsilon(\alpha)$. Hence, $\mathrm{ind}_B^G\varepsilon_1$ is a quotient of $\mathrm{ind}_K^G 1/(T-\lambda_\eta)$, where $\lambda_\eta = \lambda + \bar{\chi}_{1,\sigma}(-1) - 1$, by Proposition 3.5. The assumption on λ is translated into that $\lambda_\eta \neq 0, -1$. Hence, from (the argument of) Theorem 3.16 (2) (a), we have shown $\mathrm{ind}_K^G 1/(T-\lambda_\eta) \cong \mathrm{ind}_B^G(\chi_{\lambda_\eta+1}\otimes 1)$. We are done in this special case by twisting the character $\eta\circ\det$ back, applying Lemma 3.21.

Case 2 : $\dim\sigma > 1$.

Subcase 1 : $\bar{\chi}_{1,\sigma}(-1) = 1$

In this case, we are done, as now the assumption of *Case 2* satisfies the conditions of Proposition 3.18.

Subcase 2 : $\bar{\chi}_{1,\sigma}(-1) = -1$

Choose a non-trivial character η of k_E^1, so that $\bar{\chi}_{1,\sigma_\eta}(-1) = 1$, where $\sigma_\eta = \eta \circ \det \otimes \sigma$. There is then a non-zero G-morphism from the compact induction $\mathrm{ind}_K^G \sigma_\eta$ to the principal series $\mathrm{ind}_B^G \varepsilon_1$, where ε_1 is the character of B: $\varepsilon_1 \mid_{H_0} = \varepsilon \mid_{H_0} \cdot (\eta \circ \det)$, and $\varepsilon_1(\alpha) = \varepsilon(\alpha)$. By Proposition 3.5, such a G-morphism factors through $\mathrm{ind}_K^G \sigma_\eta/(T - \lambda_\eta)$, where λ_η is equal to $\lambda - 2$ and is non-zero by the assumption on λ in this case.

Now, we can apply Proposition 3.18; as a result, we conclude that

$$\mathrm{ind}_K^G \sigma_1/(T - \lambda_\eta) \cong \mathrm{ind}_B^G \varepsilon_1.$$

Finally, we twist both sides of the above isomorphism by the character $\eta^{-1} \circ \det$, using Lemma 3.21.

The proof of Theorem 3.1 is done.

Corollary 3.24. *An irreducible smooth representation π is supersingular if and only if π is supercuspidal.*

Proof. Let π be an irreducible smooth representation of G, with an underlying weight σ of some maximal compact open subgroup K. Suppose π is supersingular. If π is a principal series, by Proposition 3.5 and Proposition 3.7, the eigenvalue for T_σ is non-zero, which contradicts that π is supersingular. On the other side, assume π is supercuspidal. By (1) of Theorem 3.1, it factors through a quotient $\mathrm{ind}_K^G \sigma/(T_\sigma - \lambda)$, for some λ. If λ is non-zero, (2) of Theorem 3.1 implies that the former quotient representation only has principal series as its constitutes, hence one must have that $\lambda = 0$. ☐

4 Appendix A: Proof of (1) of Theorem 3.1

In this appendix, we start to prove (1) of Theorem 3.1, following the strategy in [BL94]. The major intermediate step here is to describe the right action of an Iwahori-Hecke algebra on the I_1-invariants of a compact induction, which might be of independent interest. Before doing that we remark that the statement in that theorem would follow formally if we add the condition that π is admissible, as such condition simply implies the space $\mathrm{Hom}_G(\mathrm{ind}_K^G \sigma, \pi)$ is of finite dimension, and we know that the spherical algebra $\mathcal{H}(K, \sigma)$ is a polynomial algebra in one variable over $\overline{\mathbf{F}}_p$.

4.1 The Iwahori-Hecke algebra $\mathcal{H}(I,\chi)$

For a character χ of I, the structure of the Iwahori-Hecke algebra $\mathcal{H}(I,\chi) := \mathrm{End}_G(\mathrm{ind}_I^G\chi)$, and its simple modules were determined in [KX15]. In this part we collect the results that will be used later.

We recall some conventions at first. By [BL94, Proposition 5], the algebra $\mathcal{H}(I,\chi)$ is isomorphic to the algebra $\mathcal{H}_I(\chi)$, where the latter consists of all functions φ on G with compact support and satisfying $\varphi(i_1 g i_2) = \chi(i_1 i_2)\varphi(g)$, for any $i_1, i_2 \in G$ and any $g \in G$. For an element $g \in G$, denote by φ_g the function in $\mathcal{H}_I(\chi)$ supported on IgI and satisfying $\varphi(g) = 1$. Let T_g be the operator in $\mathcal{H}(I,\chi)$ which corresponds to φ_g via the former isomorphism[3].

Proposition 4.1. (1).(Degenerate case) *When $\chi = \chi^s$, the algebra $\mathcal{H}(I,\chi)$ is noncommutative, generated by T_β and $T_{\beta'}$, both of which satisfy quadratic relations without constant terms.*

(2).(Regular case) *When $\chi \neq \chi^s$, the algebra $\mathcal{H}(I,\chi)$ is commutative, generated by the operators T_α and $T_{\alpha^{-1}}$, where their products are zero.*

4.2 The right action of $\mathcal{H}(I,\chi)$ on $(\mathrm{ind}_K^G\sigma)^{I,\chi}$

4.2.1 The (I,χ)-isotypic of $\mathrm{ind}_K^G\sigma$

Recall the Cartan-Iwahori decomposition of G:

$$G = \cup_{n\in\mathbb{Z}} K\alpha^n I.$$

Note that one may replace I by I_1 in the above decomposition.

Let σ be an irreducible smooth representation of K. A function in $\mathrm{ind}_K^G\sigma$ invariant under the action of I_1 is supported on a finite union of cosets of the form $K\alpha^k I_1$ for $k \in \mathbf{Z}$, and such an f is uniquely determined by its values on all the matrices α^k, for $k \in \mathbf{Z}$.

As the I_1-invariants of σ is one-dimensional, we fix a non-zero vector $v_0 \in \sigma^{I_1}$ and denote by χ_σ the character of I on σ^{I_1}.

Definition 4.2. (1). *When $K = K_0$, $W_K = \{Id, \beta\}$, and*

$$\omega_n = \begin{cases} \beta, & n > 0, \\ Id, & n \leq 0. \end{cases} \tag{20}$$

(2). *When $K = K_1$, $W_K = \{Id, \beta'\}$, and*

$$\omega_n = \begin{cases} Id, & n \geq 0, \\ \beta', & n < 0. \end{cases} \tag{21}$$

[3] Explicitly, T_g is determined by sending φ_{Id} to φ_g.

It is immediate from the definition that $\omega_n\alpha^{-n}I\alpha^n\omega_n \cap K \subseteq I$, for any $n \in \mathbf{Z}$.

Remark 4.3. *Sometimes we use ω_n^K for ω_n to avoid confusion. When a maximal compact open subgroup K is fixed, we denote by K' the other one and by ω' the non-trivial element in $W_{K'}$. Hence, the elements ω_n^K and $\omega_{-n}^{K'}$ are either both identity or both not, for any $n \in \mathbf{Z}$.*

Let f_n be the function in $(\mathrm{ind}_K^G\sigma)^{I_1}$, supported on $K\alpha^{-n}I_1$, such that

$$f_n(\alpha^{-n}) = \omega_n v_0.$$

Then, we have,

Proposition 4.4. (1). *The set of functions $\{f_n,\ n \in \mathbf{Z}\}$ consists of a basis of the I_1-invariant of the compact induction $\mathrm{ind}_K^G\sigma$.*

(2). *The action of the group I on function f_n is given by*

$$i \cdot f_n = \chi_\sigma^{\omega_n}(i) f_n.$$

Proof. The statement in (2) can be checked easily. For (1), let f be a non-zero I_1-invariant function in $\mathrm{ind}_K^G\sigma$, supported on the coset $K\alpha^{-n}I_1$, for some $n \in \mathbf{Z}$. For $g \in K \cap \omega_n\alpha^{-n} \cdot I_1 \cdot (\omega_n\alpha^{-n})^{-1}$, as f is I_1-invariant, $f(g \cdot \omega_n\alpha^{-n}) = \sigma(g)f(\omega_n\alpha^{-n}) = f(\omega_n\alpha^{-n})$, i.e., $f(\omega_n\alpha^{-n})$ is fixed by the action of $K \cap \omega_n\alpha^{-n} \cdot I_1 \cdot (\omega_n\alpha^{-n})^{-1}$. However, it is immediate to verify that $I_1 \subseteq K(1)(K \cap \omega_n\alpha^{-n} \cdot I_1 \cdot (\omega_n\alpha^{-n})^{-1})$. As $K(1)$ acts trivially on σ, one concludes that $f(\omega_n\alpha^{-n})$ is indeed I_1-invariant. Therefore, f only differs from f_n by a scalar. We are done. ☐

By (2) of the above Proposition, it is immediate to see the space $(\mathrm{ind}_K^G\sigma)^{I,\chi}$ is non-zero if and only if $\chi = \chi_\sigma$ or χ_σ^s.

Corollary 4.5. *For a $\omega \in W_K$, the (I, χ_σ^ω)-isotypic of the compact induction $\mathrm{ind}_K^G\sigma$ has the following set as a basis:*

(1). *When $\chi_\sigma^s = \chi_\sigma$,*

$$\{f_n; n \in \mathbf{Z}\};$$

(2). *When $\chi_\sigma^s \neq \chi_\sigma$,*

$$\{f_n; \omega_n = \omega\}.$$

4.2.2 The right action of $\mathcal{H}(I,\chi)$ on $(\mathrm{ind}_K^G\sigma)^{I,\chi}$

Lemma 4.6. (1) *If $\omega\alpha^k i\alpha^l \in I\omega\alpha^m I$ for some $i\in B\cap I$ or $B'\cap I$, then $k+l=m$;*

(2) *If $\alpha^k i\alpha^l\in I\alpha^m I$ for some $i\in B\cap I$ or $B'\cap I$, then $k+l=m$.*

Proof. This is an easy exercise. $\square$

Proposition 4.7. *Suppose $\chi=\chi^s$. Let ω be the non-trivial element in W_K. Write $\omega'=\omega\alpha^t\in W_{K'}$ for a unique $t\in\{1,-1\}$. Then we have*

(1). *For $n\in\mathbf{Z}$ such that $\omega_n=Id$:*

$$f_n\mid T_\omega=c_n\cdot f_n,\ f_n\mid T_{\omega'}=f_{-n-t};$$

(2). *For $n\in\mathbf{Z}$ such that $\omega_n=\omega$:*

$$f_n\mid T_\omega=f_{-n},\ f_n\mid T_{\omega'}=c'_n f_n.$$

Proof. First of all, recall that the right action of T_g, for an element $g\in G$, is given by:

$$f\mid T_g=\textstyle\sum_{i\in I/I\cap g^{-1}Ig} ig^{-1}\cdot f,$$

for a $f\in(\mathrm{ind}_K^G\sigma)^{I,\chi}$. When g normalizes the diagonal subgroup of I, one may identify $I/I\cap g^{-1}Ig$ with $I_1/I_1\cap g^{-1}I_1g$, and in this case a simple calculation gives that in the above sum one may assume $i\in B\cap I_1$ or $i\in B'\cap I_1$. We will use this remark in the following without mention.

Essentially we will only verify the second half of (1) and the first half of (2) in detail, where the remaining cases follow easily by applying the quadratic relations of T_ω and $T_{\omega'}$ (Proposition 4.1).

For simplicity, we treat the case $n=0$ in the first half of (1) separately. As f_0 is supported on K, $f_0\mid T_\omega$ is also supported in K. We compute then:

$$f_0\mid T_\omega(Id)=\textstyle\sum_{i\in I/I\cap\omega I\omega^{-1}} f_0(i\omega)=\sum_{i\in I/I\cap\omega I\omega^{-1}} i\omega\cdot v_0=c_0v_0,$$

where, c_0 is a constant. So we have $f_0\mid T_\omega=c_0f_0$.

We start to prove the first half of (2). In this case, the assumption is $\omega_n=\omega$. The support of the function $f_n\mid T_\omega$ is contained in $K\alpha^{-n}I\omega I$, which is a subset of $K\alpha^{-n}I\cup K\alpha^{-n}\omega I$. Hence, it suffices to evaluate the function $f_n\mid T_\omega$ at α^{-n} and α^n (as $\alpha^{-n}\omega=\omega\alpha^n$). We show firstly that $\alpha^{-n}i\omega\notin K\alpha^{-n}I=I\alpha^{-n}I\cup I\omega\alpha^{-n}I$, for any $i\in I\cap B$ or $I\cap B'$. By Lemma 4.6, it is clear that $\alpha^{-n}i\omega\notin I\omega\alpha^{-n}I$ (note that $n\neq 0$ in this case). Assume there is an $i\in I$ such that $\alpha^{-n}i\omega\in I\alpha^{-n}I$. Hence, we have $i\omega\in\alpha^nI\alpha^{-n}I\cap K$. As $\omega_n=\omega$ ($\omega_{-n}=Id$), we indeed have $\alpha^nI\alpha^{-n}I\cap K\subseteq I$. We get a contradiction, as $\omega\notin I$.

Next, we show that $\alpha^n i\omega \in K\alpha^{-n}I$ for some $i \in I$ if and only lf $i \in I \cap \omega I \omega$. The 'if' part is clear. Using Iwahori decomposition, it is immediate to see $\alpha^n i\omega \notin I\alpha^{-n}I$. Assume there is some $i \in I$ such that $\alpha^n i\omega \in I\omega\alpha^{-n}I$. We obtain that $\omega i\omega \in \omega\alpha^{-n}I\omega\alpha^{-n}I \cap K$. As $\omega_n = \omega$, the latter group is indeed contained in I. Hence, $i \in I \cap \omega I\omega$. So far, we may prove the first half of (2) as follows:

$$f_n \mid T_\omega(\alpha^n) = \textstyle\sum_{i=Id} f_n(\alpha^n i\omega) = f_n(\omega\alpha^{-n}) = \omega\omega_n v_0 = v_0,$$

hence $f_n \mid T_\omega = f_{-n}$.

We now assume $\omega_n = Id$ and proceed to prove the second half of (1). The support of the function $f_n \mid T_{\omega'}$ is contained in $K\alpha^{-n}I\omega' I$, which is a subset of $K\alpha^{-n}I \cup K\alpha^{-n-t}\omega I$. We only needs to evaluate the function at α^{-n} and α^{n+t}.

We show firstly that $\alpha^{-n}i\omega' \notin K\alpha^{-n}I$ for any $i \in I \cap B$ or $I \cap B'$. By applying Lemma 4.6, one sees easily that $\alpha^{-n}i\omega' \notin I\omega\alpha^{-n}I$. Assume there is some $i \in I$ such that $\alpha^{-n}i\omega' \in I\alpha^{-n}I$. It gives that $i\omega' \in \alpha^n I\alpha^{-n}I \cap K'$. By Remark 4.3 and $\omega_n = Id$, the latter group is indeed contained in I, which gives a contradiction as $\omega' \notin I$.

We proceed to prove that $\alpha^{n+t}i\omega' \in K\alpha^{-n}I$ if and only if $i \in I \cap \omega' I\omega'$. It suffices to show the 'only if' part. Similarly by Iwahori decomposition, it is easy to see $\alpha^{n+t}i\omega' \notin I\alpha^{-n}I$. Assume there is some $i \in I$ such that $\alpha^{n+t}i\omega' \in I\omega\alpha^{-n}I$, which gives $\omega' i\omega' \in \omega'\alpha^{-(n+t)}I\alpha^{n+t}\omega' I \cap K'$. Recall we are in the case $\omega_n^K = Id$, and one may check by definition that $\omega_{n+t}^K = Id$. By Remark 4.3, $\omega_{n+t}^{K'} = \omega'$, from which the group $\omega'\alpha^{-(n+t)}I\alpha^{n+t}\omega' I \cap K'$ is contained in I. Therefore, we have shown that $i \in I \cap \omega' I\omega'$. Now the second half of (1), i.e., $f_n \mid T_{\omega'} = f_{-n-t}$, follows easily as before.

We are done. □

In the regular case, we have the following:

Proposition 4.8. *Suppose $\chi \neq \chi^s$. For a $\omega \in W_K$, and all $n \in \mathbf{Z}$ such that $\omega_n = \omega$, there is a unique $t_\omega \in \{-1, 1\}$ such that*

$$f_n \mid T_{\alpha^{t_\omega}, \chi^\omega} = 0, \;\; f_n \mid T_{\alpha^{-t_\omega}, \chi^\omega} = f_{n+t_\omega}. \tag{22}$$

Proof. Fix a $\omega \in W_K$. Given a $t \in \{-1, 1\}$, recall the formula of $T_{\alpha^t, \chi^\omega}$:

$$f \mid T_{\alpha^t} = \textstyle\sum_{i \in I/I\cap\alpha^{-t}I\alpha^t} i\alpha^{-t} \cdot f,$$

for $f \in (\mathrm{ind}_K^G \sigma)^{I, \chi^\omega}$, where T_{α^t} is short for $T_{\alpha^t, \chi^\omega}$.

Note that f_n is supported on $K\alpha^{-n}I = I\alpha^{-n}I \cup I\omega_K\alpha^{-n}I$, where ω_K is the non-trivial element in W_K. Suppose $\alpha^k i\alpha^{-t} \in K\alpha^{-n}I$ for some $k \in \mathbf{Z}$.

From Iwahori decomposition, one sees firstly that $\alpha^k i\alpha^{-t} \notin I\omega_K\alpha^{-n}I$ for any $i \in I \cap B$ or $I \cap B'$, and any $k \in \mathbf{Z}$. If $\alpha^k i\alpha^{-t} \in I\alpha^{-n}I$, one has $k = -n + t$ by Lemma 4.6.

As we assume $\omega_n = \omega$, the function would be zero if $\omega_{n-t} \neq \omega$. It is easy to check such condition is satisfied by the unique $t \in \{-1, 1\}$ with $\omega_t = \omega$, and a unique n. Write t_ω for such t, n_ω for such n. One must then have $\omega_{n+t_\omega} = \omega$ for all n such that $\omega_n = \omega$. In fact one has $\{n : \omega_n = \omega\} = n_\omega + t_\omega\mathbb{N}_{\geq 0}$.

We proceed to prove $f_n \mid T_{\alpha^{t_\omega}} = 0$ for those n satisfying $\omega_n = \omega_{n-t_\omega} = \omega$:

$$f_n \mid T_{\alpha^{t_\omega}}(\alpha^{-n+t_\omega}) = \textstyle\sum_{i\in I/I\cap\alpha^{-t_\omega}I\alpha^{t_\omega}} f_n(\alpha^{-n+t_\omega}i\alpha^{-t_\omega}).$$

As the representatives in the sum are taken either in $B \cap I_1$ or $B' \cap I_1$, one may conclude that $\alpha^{-n+t_\omega}i\alpha^{n-t_\omega} \in K$. But as $\omega_{n-t_\omega} = \omega$, we are given $\alpha^{-n+t_\omega}i\alpha^{n-t_\omega} \in K \cap \alpha^{-n+t_\omega}I\alpha^{n-t_\omega} \subseteq \omega I\omega$. By writing $\alpha^{-n+t_\omega}i\alpha^{n-t_\omega}$ as the form $\omega i'\omega$ for some $i' \in I_1$, the above sum becomes:

$$\begin{aligned}\sum_{i\in I/I\cap\alpha^{-t_\omega}I\alpha^{t_\omega}} f_n(\omega i'\omega\alpha^{-n}) &= \sum_{i\in I/I\cap\alpha^{-t_\omega}I\alpha^{t_\omega}} \omega i'\omega\omega_n v_0 \\ &= \sum_{i\in I/I\cap\alpha^{-t_\omega}I\alpha^{t_\omega}} \omega v_0 = 0.\end{aligned}$$

So far we have proved $f_n \mid T_{\alpha^{t_\omega}} = 0$.

We proceed to prove the second half. From the discussion in the beginning, we are led to compute:

$$f_n \mid T_{\alpha^{-t_\omega}}(\alpha^{-n-t_\omega}) = \textstyle\sum_{i\in I/I\cap\alpha^{t_\omega}I\alpha^{-t_\omega}} f_n(\alpha^{-n-t_\omega}i\alpha^{t_\omega}).$$

The statement follows easily from the claim that $\alpha^{-n-t_\omega}i\alpha^{t_\omega} \in I\alpha^{-n}I$ if and only if $i \in I \cap \alpha^{t_\omega}I\alpha^{-t_\omega}$: assume there are some $i_1, i_2 \in I$ such that

$$i = \alpha^{n+t_\omega}i_1\alpha^{-n}i_2\alpha^{-t_\omega}$$

for some $i \in I \setminus I \cap \alpha^{t_\omega}I\alpha^{-t_\omega}$. Suppose i is in $B \cap I_1$ (resp, $B' \cap I_1$), then one can always assume $i_1, i_2 \in B' \cap I$ (resp, $B \cap I$), by multiplying the both sides of above identity proper elements in $B \cap I_1 \cap \alpha^{t_\omega}I_1\alpha^{-t_\omega}$(resp, $B' \cap I_1 \cap \alpha^{t_\omega}I_1\alpha^{-t_\omega}$). Note that the left side of above identity is still in $I \setminus I \cap \alpha^{t_\omega}I\alpha^{-t_\omega}$; in particular it is not the identity matrix. Now a contradiction arises as a non-trivial unipotent matrix in $B \cap I_1$ (resp, $B' \cap I_1$) can not lie in $B' \cap I$ (resp, $B \cap I$). We are done. $\square$

Based on the previous two propositions, we prove the following crucial corollary.

Corollary 4.9. *Let σ be an irreducible smooth representation of K, and χ be the character of I on σ^{I_1}. Fix an element $\omega \in W_K$. Then, any non-zero $\mathcal{H}(I,\chi^\omega)$-submodule of $(\mathrm{ind}_K^G\sigma)^{I,\chi^\omega}$ is of finite co-dimension (as a subspace).*

Proof. We deal with the regular case that $\chi \neq \chi^s$ at first. As in the proof of Proposition 4.8, let t_ω be the integer in $\{-1,1\}$ such that $\omega_{t_\omega} = \omega$, and n_ω be the unique integer such that $\omega_{n_\omega} = \omega \neq \omega_{n_\omega - t_\omega}$.

Let M be a non-zero $\mathcal{H}(I,\chi^\omega)$-submodule of $(\mathrm{ind}_K^G\sigma)^{I,\chi^\omega}$, and f be a non-zero function in M, say $f = \sum_{i=k}^{l} c_i f_i$, where k and l are integers satisfying $k \leq l$, $\omega_k = \omega_l = \omega$ and $c_k c_l \neq 0$. Replacing f by $f \mid T_{\alpha^{-t_\omega}}$ if necessary, one may assume either k or l is not equal to n_ω. To simplify the notations, we re-write $\{k,l\}$ as $\{m,m'\}$ such that $|m-n_\omega| \geq |m'-n_\omega|$.

Now let M' be the subspace of $(\mathrm{ind}_K^G\sigma)^{I,\chi^\omega}$ generated by M and the set of functions $\{f_{n_\omega}, f_{n_\omega+t_\omega}, ... f_{n_\omega+|m-t_\omega-n_\omega|t_\omega}\}$. As $c_m \neq 0$, $c_m^{-1}f$ minus a linear combination of $f_{n_\omega+t_\omega}, ..., f_{n_\omega+|m-t_\omega-n_\omega|t_\omega}$ gives $f_m \in M'$. By definition, the function $f' = f \mid T_{\alpha^{-t_\omega}} = \sum_{i=k}^{l} c_i f_{i+t_\omega}$ is still in M'. Similarly, $c_m^{-1}f'$ minus a linear combination of $f_{n_\omega+2t_\omega}, ..., f_m$ gives $f_{m+t_\omega} \in M'$. Repeating the former process, we show inductively that all the functions $f_{n_\omega+t_\omega\mathbb{N}_{\geq 0}}$ lie in M', but as mentioned before these functions consist of a basis of $(\mathrm{ind}_K^G\sigma)^{I,\chi^\omega}$. Hence $M' = (\mathrm{ind}_K^G\sigma)^{I,\chi^\omega}$ and we are done here.

We start to prove the degenerate case, using Proposition 4.7.

Recall some notations in Proposition 4.7: let ω be the non-trivial element in W_K, and write ω' as $\omega\alpha^t$ for a unique $t \in \{-1,1\}$. Assume in the following $\chi = \chi^s$. Let M be a non-zero $\mathcal{H}(I,\chi)$-submodule of $(\mathrm{ind}_K^G\sigma)^{I_1}$, and f be a non-zero function in M, say $f = \sum_{i=k}^{l} c_i f_i$, where k and l are integers satisfying $k \leq l$, and $c_k c_l \neq 0$.

We deal with a special case at first: $\omega_k = \omega_l = \omega$. Write m for the integer in $\{k,l\}$ with bigger absolute value. Let M' be the subspace of $(\mathrm{ind}_K^G\sigma)^{I,\chi}$ generated by M and the set of functions $\{f_i : i = m+t, ..., -(m+t)\}$. It is not clear in advance whether M' is also a $\mathcal{H}(I,\chi)$-submodule.

As $c_m \neq 0$, f_m is just $c_m^{-1}f$ minus a linear combination of functions in the set $\{f_i : i = m+t, ..., -(m+t)\}$, hence it is in M'. Now Proposition 4.7 gives $f \mid T_\omega = \sum_{i=k}^{l} c_i f_{-i} \in M$, from which a similar step that we have just used tells $f_{-m} \in M'$. To proceed, using Proposition 4.7 again, we see $f \mid T_\omega T_{\omega'} = \sum_{i=k}^{l} c_i f_{i-t} \in M$. So f_{m-t} is $c_m^{-1}f \mid T_\omega T_{\omega'}$ minus a linear combination of functions in the set $\{f_i : i = m, ..., -m\}$, which means that $f_{m-t} \in M'$. Similarly, by Proposition 4.7, $f \mid T_\omega T_{\omega'} T_\omega = \sum_{i=k}^{l} c_i f_{-i+t} \in M$, by subtracting from it a linear combination of functions in the set $\{f_i : i = m, ..., -m\}$, we get $f_{t-m} \in M'$. By considering the action of T_ω and $T_{\omega'}$ in turn, similar process shows that M' contains all the functions f_{m-it} and

f_{it-m} for all $i \geq 0$. This gives us $M' = (\mathrm{ind}_K^G \sigma)^{I_1}$. We are done in the special case that $\omega_k = \omega_l = \omega$.

When $\omega_k = \omega_l = Id$, we are already done by applying previous argument to the non-zero function $f \mid T_{\omega'} = \sum_{i=k}^{l} c_i f_{-i-t}$. It suffices to reduce the remaining case $\omega_k \neq \omega_l$ to previous cases. Assume $\omega_k \neq \omega_l$. For simplicity, we re-write $\{k, l\}$ as $\{m, m'\}$ such that $\omega_m = Id$ and $\omega_{m'} = \omega$. We claim $f \mid T_\omega$ and $f \mid T_{\omega'}$ can not be both zero: if $f \mid T_{\omega'} = 0$, one concludes from the formula of actions of $T_{\omega'}$ (Proposition 4.7) that $|m| < |m'|$. But in this situation, the formula of actions of T_ω (Proposition 4.7) implies that $f \mid T_\omega$ can never be zero. By looking at a non-zero function in $\{f \mid T_\omega, f \mid T_{\omega'}\}$ which is still in M, we are reduced to known cases ($\omega_k = \omega_l$). We are done. $\square$

Remark 4.10. *One might not hope a straightforward generalization of the above co-finiteness result to other groups: even for the group $SL_2(F)$ counterexamples do exist ([Abd14], Section 3.7.3]).*

4.3 Proof of (1) Theorem 3.1

In this part, based on Corollary 4.9, we prove the first part of Theorem 3.1, using an argument of Barthel–Livné ([BL94]).

Theorem 4.11. *Assume π is an irreducible smooth representation of G, containing an irreducible smooth representation σ of some maximal compact open subgroup K. Then the space*

$$\mathrm{Hom}_G(\mathrm{ind}_K^G \sigma,\ \pi)$$

has an eigenvector for the action of the Hecke algebra $\mathcal{H}(K, \sigma)$.

Proof. By assumption, we are given a non-zero K-embedding ι from σ to $\pi|_K$. Let ϕ_ι be the corresponding G-morphism in $\mathrm{Hom}_G(\mathrm{ind}_K^G \sigma,\ \pi)$ via Frobenius reciprocity.

As $\mathrm{ind}_K^G \sigma$ is not irreducible, ϕ_ι is not injective, i.e., $\mathrm{ker}\phi_\iota \neq 0$. Hence, $(\mathrm{ker}\phi_\iota)^{I_1} \neq 0$. From the description in Corollary 4.5, there is a character χ (χ_σ or χ_σ^s) such that

$$(\mathrm{ker}\phi_\iota)^{I,\,\chi} \neq 0,$$

in other words, $\mathrm{Hom}_G(\mathrm{ind}_I^G \chi,\ \mathrm{ker}\phi_\iota) \neq 0$. Denote by ϕ_ι^* the map given by the composition with ϕ_ι,

$$\phi_\iota^* :\ \mathrm{Hom}_G\,(\mathrm{ind}_I^G \chi,\ \mathrm{ind}_K^G \sigma) \to \mathrm{Hom}_G\,(\mathrm{ind}_I^G \chi,\ \pi).$$

Of course, ϕ_ι^* annihilates $\mathrm{Hom}_G(\mathrm{ind}_I^G\chi,\ \ker\phi_\iota)$, and applying Corollary 4.9 we conclude that the image of ϕ_ι^* in $\mathrm{Hom}_G\ (\mathrm{ind}_I^G\chi,\ \pi)$ is a finite dimensional $\mathcal{H}(I,\chi)$-submodule in $\mathrm{Hom}_G\ (\mathrm{ind}_I^G\chi,\ \pi)$.

For simplicity, we also denote by ϕ_ι^* the map,

$$\phi_\iota^* :\ \mathrm{Hom}_G\ (\mathrm{ind}_K^G\sigma,\ \mathrm{ind}_K^G\sigma) \to \mathrm{Hom}_G\ (\mathrm{ind}_K^G\sigma,\ \pi).$$

Let $\triangle_0$ be the K-morphism in $\mathrm{Hom}_K(\mathrm{ind}_I^K\chi,\sigma)$, corresponding to the morphism in $\mathrm{Hom}_I(\chi,\sigma)$ which maps 1 to v_0. We note that $\triangle_0$ is surjective, as σ is irreducible. Inducing these K-representations to G, we then get a G-morphism $\triangle$ in $\mathrm{Hom}_G\ (\mathrm{ind}_I^G\chi,\ \mathrm{ind}_K^G\sigma)$ from $\triangle_0$. It is also surjective ([BL94, 2.1]).

Then, $\triangle$ induces two composition maps, both denoted by $\triangle^*$:

$$\triangle^* :\ \mathrm{Hom}_G\ (\mathrm{ind}_K^G\sigma,\ \mathrm{ind}_K^G\sigma) \to \mathrm{Hom}_G\ (\mathrm{ind}_I^G\chi,\ \mathrm{ind}_K^G\sigma),$$
$$\triangle^* :\ \mathrm{Hom}_G\ (\mathrm{ind}_K^G\sigma,\ \pi) \to \mathrm{Hom}_G\ (\mathrm{ind}_I^G\chi,\ \pi).$$

Therefore, $\triangle^*$ are injective.

It is immediate from the definitions of $\triangle^*$ and ϕ_ι^* that we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_G\ (\mathrm{ind}_I^G\chi,\ \mathrm{ind}_K^G\sigma) \arrow[r, "\phi_\iota^*"] & \mathrm{Hom}_G\ (\mathrm{ind}_I^G\chi,\ \pi) \\
\mathrm{Hom}_G\ (\mathrm{ind}_K^G\sigma,\ \mathrm{ind}_K^G\sigma) \arrow[r, "\phi_\iota^*"] \arrow[u, "\triangle^*"] & \mathrm{Hom}_G\ (\mathrm{ind}_K^G\sigma,\ \pi) \arrow[u, "\triangle^*"']
\end{tikzcd}$$

In all, we conclude that $\phi_\iota^*(\mathrm{End}_G(\mathrm{ind}_K^G\sigma))$ must be a finite dimensional $\mathcal{H}(K,\sigma)$-submodule in $\mathrm{Hom}_G\ (\mathrm{ind}_K^G\sigma,\ \pi)$. As $\mathcal{H}(K,\sigma)$ is a polynomial algebra in one variable, the statement in Theorem follows. □

We record a corollary here, from which (1) of Theorem 3.1 follows.

Corollary 4.12. *Let π be an irreducible smooth representation of G, and be a quotient of some compact induction $\mathrm{ind}_K^G\sigma$, via the projection θ. Then the $\mathcal{H}(K,\sigma)$-submodule $\langle\theta\cdot\mathcal{H}(K,\sigma)\rangle$ of $\mathrm{Hom}_G\ (\mathrm{ind}_K^G\sigma,\ \pi)$ is a finite dimensional vector space.*

Proof. Immediately from the argument of last Theorem. □

Remark 4.13. *We have assumed all over the paper the coefficient field is $\overline{\mathbf{F}}_p$, but in this appendix we only need the field to be characteristic p.*

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